Intersections of Leray Complexes and Regularity of Monomial Ideals

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Abstract

For a simplicial complex X and a field $\mathbb{K}$, let $\tilde{\mathrm{h}}_i(X) = \dim \tilde{\mathrm{H}}_i(X;\mathbb{K})$. It is shown that if X, Y are complexes on the same vertex set, then for $k \geq 0$

$$\tilde{\mathrm{h}}_{k-1}(X \cap Y) \leq \sum_{\sigma \in Y} \sum_{i+j=k} \tilde{\mathrm{h}}_{i-1}(X[\sigma]) \cdot \tilde{\mathrm{h}}_{j-1}(\mathrm{lk}(Y,\sigma)) \quad .$$

A simplicial complex X is *d-Leray* over $\mathbb{K}$, if $\tilde{\mathrm{H}}_i(Y;\mathbb{K}) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $L_{\mathbb{K}}(X)$ denote the minimal d such that X is d-Leray over $\mathbb{K}$. The above theorem implies that if X, Y are simplicial complexes on the same vertex set then

$$L_{\mathbb{K}}(X \cap Y) \leq L_{\mathbb{K}}(X) + L_{\mathbb{K}}(Y) \quad .$$

Reformulating this inequality in commutative algebra terms, we obtain the following result conjectured by Terai: If I, J are square-free monomial ideals in $S = \mathbb{K}[x_1, \ldots, x_n]$, then

$$\mathrm{reg}(I + J) \leq \mathrm{reg}(I) + \mathrm{reg}(J) - 1$$

where $\mathrm{reg}(I)$ denotes the Castelnuovo-Mumford regularity of I.

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1 Introduction

Let X be a simplicial complex on the vertex set V. The *induced* subcomplex on a subset of vertices $S \subset V$ is $X[S] = \{\sigma \in X : \sigma \subset S\}$. Let $\{\}$ be the *void complex* and let $\{\emptyset\}$ be the *empty complex*. Any non-void complex contains $\emptyset$ as a unique (-1)-dimensional face. The *star* of a subset $A \subset V$ is $\mathrm{St}(X, A) = \{\tau \in X : \tau \cup A \in X\}$. The *link* of $A \subset V$ is $\mathrm{lk}(X, A) = \{\tau \in \mathrm{St}(X, A) : \tau \cap A = \emptyset\}$. If $A \notin X$ then $\mathrm{St}(X, A) = \mathrm{lk}(X, A) = \{\}$. All homology groups considered below are with coefficients in a fixed field $\mathbb{K}$ and we denote $\tilde{\mathrm{h}}_i(X) = \dim_{\mathbb{K}} \tilde{\mathrm{H}}_i(X)$. Note that $\tilde{\mathrm{h}}_{-1}(\{\}) = 0 \neq 1 = \tilde{\mathrm{h}}_{-1}(\{\emptyset\})$. Our main result is the following

Theorem 1.1. *Let X, Y be finite simplicial complexes on the same vertex set. Then for $k \geq 0$*

$$\tilde{\mathrm{h}}_{k-1}(X \cap Y) \leq \sum_{\sigma \in Y} \sum_{i+j=k} \tilde{\mathrm{h}}_{i-1}(X[\sigma]) \cdot \tilde{\mathrm{h}}_{j-1}(\mathrm{lk}(Y, \sigma)) \quad . \tag{1}$$

We next discuss some applications of Theorem 1.1. A simplicial complex X is *d-Leray* over $\mathbb{K}$ if $\tilde{\mathrm{H}}_i(Y) = 0$ for all induced subcomplexes $Y \subset X$ and $i \geq d$. Let $\mathrm{L}_{\mathbb{K}}(X)$ denote the minimal d such that X is d-Leray over $\mathbb{K}$. Note that $\mathrm{L}_{\mathbb{K}}(X) = 0$ iff X is a simplex. $\mathrm{L}_{\mathbb{K}}(X) \leq 1$ iff X is the clique complex of a chordal graph (see e.g. [11]).

The class $\mathcal{L}_{\mathbb{K}}^d$ of d-Leray complexes over $\mathbb{K}$ arises naturally in the context of Helly type theorems [3]. The *Helly number* $\mathrm{h}(\mathcal{F})$ of a finite family of sets $\mathcal{F}$ is the minimal positive integer h such that if $\mathcal{K} \subset \mathcal{F}$ satisfies $\bigcap_{K \in \mathcal{K}'} K \neq \emptyset$ for all $\mathcal{K}' \subset \mathcal{K}$ of cardinality $\leq h$, then $\bigcap_{K \in \mathcal{K}} K \neq \emptyset$. The *nerve* $N(\mathcal{K})$ of a family of sets $\mathcal{K}$, is the simplicial complex whose vertex set is $\mathcal{K}$ and whose simplices are all $\mathcal{K}' \subset \mathcal{K}$ such that $\bigcap_{K \in \mathcal{K}'} K \neq \emptyset$. It is easy to see that for any field $\mathbb{K}$

$$\mathrm{h}(\mathcal{F}) \leq 1 + \mathrm{L}_{\mathbb{K}}(N(\mathcal{F})).$$

For example, if $\mathcal{F}$ is a finite family of convex sets in $\mathbb{R}^d$, then by the Nerve Lemma (see e.g. [2]) $N(\mathcal{F})$ is d-Leray over $\mathbb{K}$, hence follows Helly's Theorem: $\mathrm{h}(\mathcal{F}) \leq d+1$. This argument actually proves the Topological Helly Theorem: If $\mathcal{F}$ is a finite family of closed sets in $\mathbb{R}^d$ such that the intersection of any subfamily of $\mathcal{F}$ is either empty or contractible, then $\mathrm{h}(\mathcal{F}) \leq d + 1$.

Nerves of families of convex sets however satisfy a stronger combinatorial property called *d-collapsibility* [11], that leads to some of the deeper extensions of Helly's Theorem. It is of considerable interest to understand which combinatorial properties of nerves of families of convex sets in $\mathbb{R}^d$ extend to arbitrary d-Leray complexes. For some recent work in this direction see [1, 6]. One consequence of Theorem 1.1 is the following

Theorem 1.2. *Let $X_1, \ldots, X_r$ be simplicial complexes on the same finite vertex set. Then*

$$\mathrm{L}_{\mathbb{K}}\big(\bigcap_{i=1}^{r} X_i\big) \leq \sum_{i=1}^{r} \mathrm{L}_{\mathbb{K}}(X_i) \tag{2}$$

$$\mathrm{L}_{\mathbb{K}}\big(\bigcup_{i=1}^{r} X_i\big) \leq \sum_{i=1}^{r} \mathrm{L}_{\mathbb{K}}(X_i) + r - 1 \quad . \tag{3}$$

Example: Let $V_1, \ldots, V_r$ be disjoint sets of cardinalities $|V_i| = a_i$, and let $V = \bigcup_{i=1}^{r} V_i$. Let $\Delta(A)$ denote the simplex on vertex set A, with boundary $\partial\Delta(A) \simeq S^{|A|-2}$. Consider the complexes

$$X_i = \Delta(V_1) * \cdots * \Delta(V_{i-1}) * \partial\Delta(V_i) * \Delta(V_{i+1}) * \cdots * \Delta(V_r) \quad .$$

Then

$$\bigcap_{i=1}^{r} X_i = \partial\Delta(V_1) * \cdots * \partial\Delta(V_r) \simeq S^{\sum_{i=1}^{r} a_i - r - 1}$$

and

$$\bigcup_{i=1}^{r} X_i = \partial\Delta(V_1 \cup \ldots \cup V_r) \simeq S^{\sum_{i=1}^{r} a_i - 2} \quad .$$

The only non-contractible induced subcomplex of X_i is $\partial\Delta(V_i)$, therefore $\mathrm{L}_{\mathbb{K}}(X_i) = a_i - 1$. Similar considerations show that $\mathrm{L}_{\mathbb{K}}(\cup_{i=1}^{r} X_i) = \sum_{i=1}^{r} a_i - 1$ and $\mathrm{L}_{\mathbb{K}}(\cap_{i=1}^{r} X_i) = \sum_{i=1}^{r} a_i - r$, so equality is attained in both (2) and (3).

Theorem 1.2 was first conjectured in a different but equivalent form by Terai [8], in the context of monomial ideals . Let $S = \mathbb{K}[x_1, \ldots, x_n]$ and let M be a graded S-module. Let $\beta_{ij}(M) = \dim_{\mathbb{K}} \mathrm{Tor}_i^S(\mathbb{K}, M)_j$ be the graded Betti numbers of M. The *regularity* of M is the minimal $\rho = \mathrm{reg}(M)$ such that $\beta_{ij}(M)$ vanish for $j > i + \rho$ (see e.g. [4]).
For a simplicial complex X on $[n] = \{1, \ldots, n\}$ let I_X denote the ideal of S generated by $\{\prod_{i \in A} x_i : A \notin X\}$. The following fundamental result of Hochster relates the Betti numbers of I_X to the topology of the induced subcomplexes X.

Theorem 1.3 (Hochster [5])**.**

$$\beta_{ij}(I_X) = \sum_{|W|=j} \dim_{\mathbb{K}} \tilde{\mathrm{H}}_{j-i-2}(X[W]) \quad . \tag{4}$$

Hochster's formula (4) implies that $\mathrm{reg}(I_X) = \mathrm{L}_{\mathbb{K}}(X) + 1$. The case $r = 2$ of Theorem 1.2 is therefore equivalent to the following result conjectured by Terai [8].

Theorem 1.4. *Let X and Y be simplicial complexes on the same vertex set. Then*

$$\mathrm{reg}(I_X + I_Y) = \mathrm{reg}(I_{X\cap Y}) \leq \mathrm{reg}(I_X) + \mathrm{reg}(I_Y) - 1$$
$$\mathrm{reg}(I_X \cap I_Y) = \mathrm{reg}(I_{X\cup Y}) \leq \mathrm{reg}(I_X) + \mathrm{reg}(I_Y) \quad .$$

□

Theorem 1.4 can also be formulated in terms of projective dimension. Let $X^* = \{\tau \subset [n] : [n] - \tau \not\in X\}$ denote the Alexander dual of X. Terai [7] showed that

$$\mathrm{pd}(S/I_X) = \mathrm{reg}(I_{X^*}) \quad . \tag{5}$$

Using (5) it is straightforward to check that Theorem 1.4 is equivalent to

Theorem 1.5.

$$\mathrm{pd}(I_X \cap I_Y) \leq \mathrm{pd}(I_X) + \mathrm{pd}(I_Y)$$
$$\mathrm{pd}(I_X + I_Y) \leq \mathrm{pd}(I_X) + \mathrm{pd}(I_Y) + 1 \quad .$$

□

In Section 2 we give a spectral sequence for the relative homology group $\mathrm{H}_*(Y, X \cap Y)$, which directly implies Theorem 1.1. The proof of Theorem 1.2 is given in Section 3.

2 A Spectral Sequence for $\mathrm{H}_*(Y, X \cap Y)$

Let K be a simplicial complex. The subdivision $\mathrm{sd}(K)$ is the order complex of the set of the non-empty simplices of K ordered by inclusion. For $\sigma \in K$ let $D_K(\sigma)$ denote the order complex of the interval $[\sigma, \cdot] = \{\tau \in K \ : \ \tau \supset \sigma\}$.

$D_K(\sigma)$ is called the *dual cell* of σ. Let $\dot{D}_K(\sigma)$ denote the order complex of the interval $(\sigma, \cdot] = \{\tau \in K \ : \ \tau \supsetneqq \sigma\}$. Note that $\dot{D}_K(\sigma)$ is isomorphic to $\mathrm{sd}(\mathrm{lk}(K,\sigma))$ via the simplicial map $\tau \to \tau - \sigma$. Since $D_K(\sigma)$ is contractible, it follows that $\mathrm{H}_i(D_K(\sigma), \dot{D}_K(\sigma)) \cong \tilde{\mathrm{H}}_{i-1}(\mathrm{lk}(K,\sigma))$ for all $i \geq 0$. Write $K(p)$ for the family of p-dimensional simplices in K. The proof of Theorem 1.1 depends on the following

Proposition 2.1. *Let X and Y be two complexes on the same vertex set V, such that $\dim Y = n$. Then there exists a homology spectral sequence $\{E^r_{p,q}\}$ converging to $\mathrm{H}_*(Y, X \cap Y)$ such that*

$$E^1_{p,q} = \bigoplus_{\sigma \in Y(n-p)} \bigoplus_{\substack{i,j \geq 0 \\ i+j=p+q}} \tilde{\mathrm{H}}_{i-1}(X[\sigma]) \otimes \tilde{\mathrm{H}}_{j-1}(\mathrm{lk}(Y,\sigma))$$

for $0 \leq p \leq n$, $0 \leq q$, *and* $E^1_{p,q} = 0$ *otherwise.*

Proof: Let Δ denote the simplex on V. For $0 \leq p \leq n$ let

$$K_p = \bigcup_{\substack{\sigma \in Y \\ \dim \sigma \geq n-p}} \Delta[\sigma] \times D_Y(\sigma) \subset Y \times \mathrm{sd}(Y)$$

and

$$L_p = \bigcup_{\substack{\sigma \in Y \\ \dim \sigma \geq n-p}} X[\sigma] \times D_Y(\sigma) \subset (X \cap Y) \times \mathrm{sd}(Y) \quad .$$

Write $K = K_n$, $L = L_n$. Let

$$\pi : K \to \bigcup_{\sigma \in Y} \Delta[\sigma] = Y$$

denote the projection on the first coordinate. For a point $z \in Y$, let $\tau = \mathrm{supp}(z)$ denote the minimal simplex in Y containing z. The fiber $\pi^{-1}(z) = \{z\} \times D_Y(\tau)$ is a cone, hence π is a homotopy equivalence. Similarly, the restriction

$$\pi_{|L} : L \to \bigcup_{\sigma \in Y} X[\sigma] = X \cap Y$$

is a homotopy equivalence. Let $F_p = C_*(K_p, L_p)$ be the group of cellular chains of the pair (K_p, L_p). The filtration $0 \subset F_0 \subset \cdots \subset F_n = C_*(K, L)$

gives rise to a homology spectral sequence $\{E^r\}$ converging to $\mathrm{H}_*(K,L) \cong \mathrm{H}_*(Y, X\cap Y)$. We compute E^1 by excision and the Künneth formula:

$$E^1_{p,q} = \mathrm{H}_{p+q}(F_p/F_{p-1}) \cong \mathrm{H}_{p+q}(K_p, L_p \cup K_{p-1}) \cong$$

$$\mathrm{H}_{p+q}\Big(\bigcup_{\sigma\in Y(n-p)} \Delta[\sigma]\times D_Y(\sigma), \bigcup_{\sigma\in Y(n-p)} X[\sigma]\times D_Y(\sigma) \;\cup\; \Delta[\sigma]\times \dot{D}_Y(\sigma) \Big) \;\cong$$

$$\bigoplus_{\sigma\in Y(n-p)} \mathrm{H}_{p+q}\big(\Delta[\sigma]\times D_Y(\sigma), X[\sigma]\times D_Y(\sigma) \;\cup\; \Delta[\sigma]\times \dot{D}_Y(\sigma)\big) \cong$$

$$\bigoplus_{\sigma\in Y(n-p)} \bigoplus_{i+j=p+q} \mathrm{H}_i(\Delta[\sigma], X[\sigma]) \otimes \mathrm{H}_j(D_Y(\sigma), \dot{D}_Y(\sigma)) \cong$$

$$\bigoplus_{\sigma\in Y(n-p)} \bigoplus_{i+j=p+q} \tilde{\mathrm{H}}_{i-1}(X[\sigma]) \otimes \tilde{\mathrm{H}}_{j-1}(\mathrm{lk}(Y,\sigma)) \quad .$$

□

Remark: The derivation of the above spectral sequence may be viewed as a simple application of the method of simplicial resolutions. See Vassiliev's papers [9, 10] for a description of this technique, and for far reaching applications to plane arrangements and to spaces of Hermitian operators.

Proof of Theorem 1.1: By Proposition 2.1

$$\tilde{\mathrm{h}}_{k-1}(X\cap Y) \le \tilde{\mathrm{h}}_{k-1}(Y) + \mathrm{h}_k(Y, X\cap Y) \le$$

$$\tilde{\mathrm{h}}_{k-1}(Y) + \sum_{p+q=k} \dim E^1_{p,q} =$$

$$\tilde{\mathrm{h}}_{k-1}(Y) + \sum_{\substack{\emptyset\ne\sigma\in Y \\ \dim\sigma\ge n-k}} \sum_{i+j=k} \tilde{\mathrm{h}}_{i-1}(X[\sigma]) \cdot \tilde{\mathrm{h}}_{j-1}(\mathrm{lk}(Y,\sigma)) \le$$

$$\sum_{\sigma\in Y} \sum_{i+j=k} \tilde{\mathrm{h}}_{i-1}(X[\sigma]) \cdot \tilde{\mathrm{h}}_{j-1}(\mathrm{lk}(Y,\sigma)) \quad .$$

□

3 Intersection of Leray Complexes

We first recall a well-known characterization of d-Leray complexes. For completeness we include a proof.

Proposition 3.1. *For a simplicial complex X, the following conditions are equivalent:*

(i) X *is d-Leray over* $\mathbb{K}$.

(ii) $\tilde{\mathrm{H}}_i(\mathrm{lk}(X,\sigma)) = 0$ *for every* $\sigma \in X$ *and* $i \geq d$.

It will be convenient to prove a slightly more general result. Let $k, m \geq 0$. We say that a simplicial complex X on V satisfies *condition* $P(k,m)$ if $\tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A],B)\big) = 0$ for all $B \subset A \subset V$ such that $|A| \geq |V| - k$, $|B| \leq m$.

Claim 3.2. *If $k \geq 0$ and $m \geq 1$ then conditions $P(k,m)$ and $P(k+1,m-1)$ are equivalent.*

Proof: Suppose $B \subset A \subset V$ and $B_1 \subset A_1 \subset V$ satisfy $B = B_1 \cup \{v\}$, $A = A_1 \cup \{v\}$ for some $v \notin A_1$, and let

$$Z_1 = \mathrm{lk}(X[A_1], B_1) \quad , \quad Z_2 = \mathrm{St}(\mathrm{lk}(X[A], B_1), v) \ .$$

Then

$$Z_1 \cup Z_2 = \mathrm{lk}(X[A], B_1) \quad , \quad Z_1 \cap Z_2 = \mathrm{lk}(X[A], B)$$

and by Mayer-Vietoris there is an exact sequence

$$\begin{aligned} \ldots \to \tilde{\mathrm{H}}_{i+1}\big(\mathrm{lk}(X[A],B_1)\big) &\to \tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A],B)\big) \to \\ \tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A_1],B_1)\big) &\to \tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A],B_1)\big) \to \ldots \quad . \end{aligned} \tag{6}$$

$\mathbf{P(k,m) \Rightarrow P(k+1,m-1)}$: Suppose X satisfies $P(k,m)$ and let $B_1 \subset A_1 \subset V$ such that $|V| - |A_1| = k+1$ and $|B_1| \leq m-1$. Choose a $v \in V - A_1$ and let $A = A_1 \cup \{v\}$, $B = B_1 \cup \{v\}$. Let $i \geq d$, then by the assumption on X, both the second and the fourth terms in (6) vanish. It follows that $\tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A_1],B_1)\big) = 0$ as required.
$\mathbf{P(k+1,m-1) \Rightarrow P(k,m)}$: Suppose X satisfies $P(k+1,m-1)$ and let $B \subset A \subset V$ such that $|V| - |A| \leq k$ and $|B| = m$. Choose a $v \in B$ and let $A_1 = A - v$, $B_1 = B - v$. Let $i \geq d$, then by the assumption on X, both the first and the third terms in (6) vanish. It follows that $\tilde{\mathrm{H}}_i\big(\mathrm{lk}(X[A],B)\big) = 0$ as required.

$\square$

Proof of Proposition 3.1: Let X be a complex on n vertices. Then (i) is equivalent to $P(n,0)$, while (ii) is equivalent to $P(0,n)$. On the otherhand, $P(n,0)$ and $P(0,n)$ are equivalent by Claim 3.2.

$\square$

Proof of Theorem 1.2: By induction it suffices to consider the $r = 2$ case. Let X, Y be complexes on V with $\mathrm{L}_{\mathbb{K}}(X) = a$, $\mathrm{L}_{\mathbb{K}}(Y) = b$, and let $k > a + b$. Then for any $\sigma \in Y$ and for any i, j such that $i + j = k$, either $i > a$ hence $\tilde{\mathrm{h}}_{i-1}(X[\sigma]) = 0$, or $j > b$ which by Proposition 3.1 implies that $\tilde{\mathrm{h}}_{j-1}(\mathrm{lk}(Y, \sigma)) = 0$. By Theorem 1.1 it then follows that $\tilde{\mathrm{h}}_{k-1}(X \cap Y) = 0$. Therefore

$$\mathrm{L}_{\mathbb{K}}(X \cap Y) \leq \max_{S \subset V} \big(\mathrm{L}_{\mathbb{K}}(X[S]) + \mathrm{L}_{\mathbb{K}}(Y[S]) \big) = \mathrm{L}_{\mathbb{K}}(X) + \mathrm{L}_{\mathbb{K}}(Y) \quad . \tag{7}$$

Next, let $k \geq \mathrm{L}_{\mathbb{K}}(X) + \mathrm{L}_{\mathbb{K}}(Y) + 1$. Then by (7) and the Mayer-Vietoris sequence

$$\to \tilde{\mathrm{H}}_k(X) \oplus \tilde{\mathrm{H}}_k(Y) \to \tilde{\mathrm{H}}_k(X \cup Y) \to \tilde{\mathrm{H}}_{k-1}(X \cap Y) \to$$

it follows that $\tilde{\mathrm{H}}_k(X \cup Y) = 0$. Hence

$$\mathrm{L}_{\mathbb{K}}(X \cup Y) \leq \max_{S \subset V} \big(\mathrm{L}_{\mathbb{K}}(X[S]) + \mathrm{L}_{\mathbb{K}}(Y[S]) + 1 \big) = \mathrm{L}_{\mathbb{K}}(X) + \mathrm{L}_{\mathbb{K}}(Y) + 1.$$

$\square$

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