Is the Universe Noise Sensitive?

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1 Noise sensitivity

Noise sensitivity is a notion related to probability and statistical physics that came up in my work with Itai Benjamini and Oded Schramm [3], which introduced this notion and mainly studies the model of percolation. A similar notion arose in the work of Tsirelson and Vershik [18], whose motivation came from mathematical quantum physics and the construction of "non-Fock spaces." Noise sensitivity and the related notions of "non-classical stochastic processes" and "black noise" are further studied and applied to mathematical physics, theoretical computer science, social choice theory, and other areas, e.g., in [15, 10, 12, 14]. The notion of noise sensitivity applies both to classical and quantum stochastic models; see [21]. An implicit motivation for Tsirelson and Vershik's paper was the idea that the Big Bang could be a natural occurrence of black noise. For an early high-energy physics non-Fock "toy model" see [19].

Noise sensitivity is related to some earlier works [9, 2, 8, 7], which study "harmonic analysis over the group $Z/2$" of certain stochastic processes arising in combinatorics, computer science, and mathematical physics. Here, Z/2 refers to the group of two elements.

Let me briefly describe the phenomenon of "noise sensitivity." When you look at the spectral decomposition of various functions related to statistical physics models (like percolation) you discover that a substantial amount (or even most) of the "energy" is concentrated on eigenfunctions such that the eigenvalues are unbounded; namely, they depend on some parameter of the system that goes to infinity in the limit. The "primal" equivalent description asserts that these functions are extremely sensitive to small stochastic perturbation of the variables. For noise-sensitive models based on geometric lattice models, the eigenfunctions which support their "energy" are interesting geometric stochastic objects, leading to interesting scaling limits, and related to critical exponents.

We refer (informally) to a stochastic process that can be regarded as the limit of stochastic functions f_n defined on ever finer lattice models as t-noise sensitive if in this representation the amount of energy on bounded eigenvalues is $1-t$ of the total energy. If $t=0$ we refer to the process as noise stable (or classical). The case where $t=1$, that is, an (asymptotically) complete noise sensitivity, appears in various examples, some going back to [5], and it is forced in certain cases by symmetry [7, 8]. Noise sensitivity (surprisingly) occurs in percolation [3, 14, 13], first-passage percolation [4], and the distribution of the largest eigenvalues of random matrices [17, 11]. Benjamini, Kalai, and Schramm showed [3, 4] that noise sensitivity necessarily emerges in very general circumstances.

Now, if you replace $Z/2$ by a fixed group Γ like $Z/3$, $U(1)$, or $SU(2)$ (or, more generally, consider products of a fixed graph or space), the basic notions and various results still extend, but there are some phenomena and new questions. (See, e.g., [6, 1].) Of interest is the study of "noise sensitivity" for harmonic analysis based on representations of a fixed non-Abelian group, as well as, more refined notions that take into accounts the type of representations that occur. It is also interesting to study spectral decomposition and noise sensitivity for probability distributions described by Potts and related models of interacting particles including analogous $O(n)$-models.

2 The universe

My very crude picture of the physicists' view of the universe (taken mainly from popular accounts) in terms of particles corresponding to specific low-eigenvalue representations and some essentially pairwise correlations/interactions between them, corresponds to what we refer to as a "noise-stable" stochastic process. (The representations involved are of some fixed groups, be they $U(1)$ for electromagnetism, or $U(1) \times SU(2) \times SU(3)$ for the "standard model," or larger but fixed groups for more general theories.) Recall from Section 1 that there are richer forms of stochastic processes where the picture is very different: much "energy" is concentrated on very "high" eigenvalues with eigenfunctions that correspond to "large" stochastic geometric objects.

Is it possible that our universe is t-noise sensitive for some t, $0 < t < 1$, that is, when described by a limit of discrete models does it have a substantial amount (a t-proportion) of "energy" on unbounded high eigenvalues? Such a possibility might be of no consequence for the noise-stable part describing the properties of particles and their interactions. Here are some possible (naive) related questions:

- Is noise sensitivity related to unexplained notions of energy and mass, e.g., dark mass and dark energy?
- Is it indeed the case that the basic current models of particle physics are noise stable? (Or is there an internal inconsistency about their noise sensitivity?)
- Would the noise sensitivity of the universe be of relevance regarding mathematical foundations of QED/QCD?
- Do noise-sensitive (black) stochastic perturbations of classical PDE appearing in physics have interesting or desirable properties? (Compare [20], Section 8.2.)
- Is noise sensitivity related to old ideas from physics on energy/mass not carried by particles. (Even obsolete or abandoned ideas can still be related to interesting mathematics.)
- Suppose the universe is t-noise sensitive for some fraction t. Would this allow for string (or string-like) theory to exist in lower dimensions? In 3+1 dimensions?

It is important to point out that the definitions of the noise-sensitivity/noise-stability dichotomy require some presentation via i.i.d. variables. To make the questions about physics rigorous, extensions of the notion of noise sensitivity are required. (Otherwise, we can restrict our attention to special cases from physics where the original definitions apply.) Intuitively, for the general case, noise sensitivity describes a situation where a stochastic process cannot be described or well approximated by statistics on a bounded number of elements. Finding the right general mathematical formulation is interesting in its own right.

Of course, the main point is this: if noise stability is an implicit assumption made in current physics models for high-energy physics, and if noise sensitivity is a possibility, then this may enable us to move forward in problems where current models are insufficient. If noise stability is a law of physics or a (rather strong) consequence of current laws of physics, this is interesting as well.

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