Three Theorems, with Computer-Aided Proofs, on Three-Dimensional Faces and Quotients of Polytopes

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Abstract

We prove that there is a finite list of 3-polytopes so that every rational d-polytope, $d \geq 9$, contains a 3-dimensional face in the list. A similar result where "faces" are replaced by "quotients" is proved already for (general) 5-polytopes. We also prove that every d-polytope, $d \geq 9$, contains a 3-dimensional quotient which is a simplex.

1 Introduction

Theorem 1 *There is a finite list of 3-dimensional polytopes such that every rational 9-polytope contains a 3-dimensional face in the list.*

Here, a rational polytope is a polytope whose vertices have rational coordinates. Duals of neighborly 4-polytopes show that the dimension 9 cannot be reduced to 4.

Theorem 2 *Every 9-dimensional polytope has the 3-dimensional simplex as a quotient.*

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The 24-cell shows that, here too, the dimension 9 cannot be reduced to 4.

The analogous situation for 2-dimensional faces and quotients is classic. It follows easily from Euler's theorem that every 3-polytope (and hence every higher-dimensional polytope) has a 2-dimensional face which is a triangle, quadrangle or a pentagon. In fact, this result (in a dual form) can be traced back to the writings of Descartes on polyhedra, see [4]. It also follows easily from Euler's theorem that every 3-polytope or its dual contains a triangular face and thus every d-polytope, $d \geq 3$ contains a triangle as a quotient.

Theorems 1 and 2 are special cases of some far-reaching conjectures, see [11]: Define for every $k > 1$ four functions $d_1 = d_1(k)$, $d_2 = d_2(k)$, $d_3 = d_3(k)$ and $d_4 = d_4(k)$ to be the smallest integers so that

- There is a finite list of k-polytopes so that every d-polytope, $d \geq d_1(k)$, has a k-dimensional quotient in the list.
- There is a finite list of k-polytopes so that every d-polytope, $d \geq d_2(k)$, contains a k-face in the list.
- Every d-polytope, $d \geq d_3(k)$, has a simplex as a k-dimensional quotient.
- Every d-polytope, $d \geq d_4(k)$, contains a k-face which is combinatorially isomorphic to a simplex or cube.

It is conjectured that all these four functions are finite. This will be easiest to prove for d_1 and hardest for d_4. Clearly, $d_1(k) \leq d_3(k) \leq d_4(k+1)$ and $d_1(k) \leq d_2(k) \leq d_4(k)$. The following Theorem gives that $d_1(3) \leq 5$. We do not know if the correct value for $d_1(3)$ is 4 or 5. We can prove also that $d_1(4) \leq 7$.

Theorem 3 *Every 5-polytope contains a 3-dimensional quotient with at most 8 vertices.*

The proofs of Theorems 1,2 and 3 as a consequence of (rather deep) known inequalities for flag numbers of polytopes were achieved by a computerized program FLAGTOOL, see [12, 14]. A comprehensive description of FLAGTOOL and theorems proved by FLAGTOOL has appeared in [14].

This work is closely related to various ideas and results by Branko Grünbaum. The proof uses the lower bound theorem for polytopes and some of its far-reaching generalizations. Some of these generalizations are based on the rigidity theory for polytopes, a topic of "lost mathematics" Grünbaum helped to revive [7]. The type of reasoning (convolutions) used here also has roots in some early papers of Grünbaum. But more than that, we feel these proofs touch on some fundamental issues concerning mathematical interest, elegance in mathematics and the use of computers raised by Grünbaum in various places see [6, 8].

2 Face numbers, flag numbers, g-numbers and convolutions

For a d-polytope P the number of k-faces is denoted by $f_k(P)$. (We will also use the notation f_k^d unless the value of d will be clear from the context.) The vector $(f_0(P), f_1(P), \dots f_{d-1}(P))$ is called the f-vector of P. For a subset $S = \{i_1, \dots, i_k\} \subset \{0, 1, \dots, d-1\}$ the flag number $f_S(P)$ is the number of chains $F_1 \subset F_2 \subset \cdots \subset F_k$ of faces of P such that $dim F_j = i_j$. (Again, we will use also the notation f_S^d.) A remarkable theorem of Bayer and Billera asserts that the affine dimension of the space of flag numbers of d-polytopes is $c_d - 1$, where c_d is the d-th Fibonacci number. Bayer and Billera showed that every flag number f_S^d can be expressed as a linear combination of *special* flag numbers f_T^d, where $T \subset \{0, 1, \dots, d-2\}$ and T contains no two consecutive integers. Their argument relies only on Euler's formula (for arbitrary dimension) and it therefore applies not only for polytopes but for arbitrary *Eulerian* posets, see [17].

Certain linear combinations of face numbers of simplicial polytopes called h-numbers and g-numbers play a crucial role in the combinatorial theory of simplicial polytopes, see [13, 15]. Intersection homology theory has led to deep and mysterious extensions of h- and g-numbers from simplicial polytopes to general polytopes. The definition (which can be found also in [16]) goes as follows. For a polytope P denote by P_k the set of k-faces of P.

Define by induction two polynomials

$$h_P(x) = \sum_{k=0}^{d} h_k^d x^{d-k}, \quad g_P(x) = \sum_{k=0}^{[d/2]} g_k^d x^{d-k},$$

by the rules: (a) $g^d_k = h^d_k - h^d_{k-1}$, (b) If P is the empty polytope or a 0-polytope P, $h_P = g_P = 1$, and

$$h_P(x) = \sum_{k=0}^{d} (x-1)^{d-k} \sum \{g_F(x) : x \in P_k\}.$$

(If the value of d is clear from the context the superscripts of h^d_k are omitted.)

Thus $g^d_1(P) = f_0(P) - d - 1$ and

$$g^d_2(P) = f_1(P) + \sum\{f_0(F) - 3 : F \in P_2\} - df_0(P) + \binom{d+1}{2}.$$

The value of g^d_2 for general d-polytopes has a rigidity theoretic meaning [9] and is nonnegative for every polytope. This extends the famous lower bound theorem of Barnette [1]. The nonnegativity of g^d_2 is still open for more general objects like polyhedral $(d-1)$-spheres and manifolds. It follows from intersection homology theory for toric varieties that the g^d_k are nonnegative for every rational d-polytope. This is still open in the non-rational case. For a d-polytope P we denote $\bar{g}^d_k(P) = g^d_k(P^*)$, where P is the dual polytope to P.

Let m^d, m^e be linear combinations of flag numbers of $d-$ and $e-$polytopes respectively. For a polytope P of dimension $d + e + 1$ define the convolution of m^d and m^e by $m^d * m^e(P) = \sum\{m^d(F) \cdot m^e(P/F) : \text{F a } d\text{-face of } P\}$.

The following lemma [10] is immediate

Lemma 4 *(1) $m^d * m^e(P)$ is a linear combination of flag numbers of $(d+e+1)$-polytopes.*

(2) If $m^d(P) = 0$ for every d-polytope P or $m^e(Q) = 0$ for every e-polytope Q then $m^d * m^e(R) = 0$ for every $(d+e+1)$-polytope R.

(3) If $m^d(P) \geq 0$ for every d-polytope P and $m^e(Q) \geq 0$ for every e-polytope Q then $m^d * m^e(R) \geq 0$ for every $(d+e+1)$-polytope R.

3 FLAGTOOL

FLAGTOOL [14, 12] is a computer program that

- computes all (known) linear relations between the flag numbers of general d-polytopes for small dimensions, $3 \leq d \leq 10$,

- extracts and automatically proves new results from those relations.

Since every flag number can be expressed as an affine combination of special flag numbers, all inequalities generated by the program will be expressed in terms of special flag numbers. See Figure 1 for the general scheme of supporting theorem proving with FLAGTOOL.

Theorem 1 is a consequence of the following stronger statement.

Theorem 5 *Every rational d-polytope ($d \geq 9$) has a 3-face with less than 78 vertices or 78 facets.*

Proof: Assume that every 3-face of an arbitrary 9-polytope has 78 or more vertices or facets. This assumption can be expressed by the inequalities $f_0^3 - 78 \geq 0$ and $f_2^3 - 78 \geq 0$. The following system of 53 linear 9-forms obtained by convolutions of the g-numbers and of these two added inequalities (in the bottom interval $[-1, 3]$) has no nonnegative feasible solution and therefore Theorem 1 is proved.

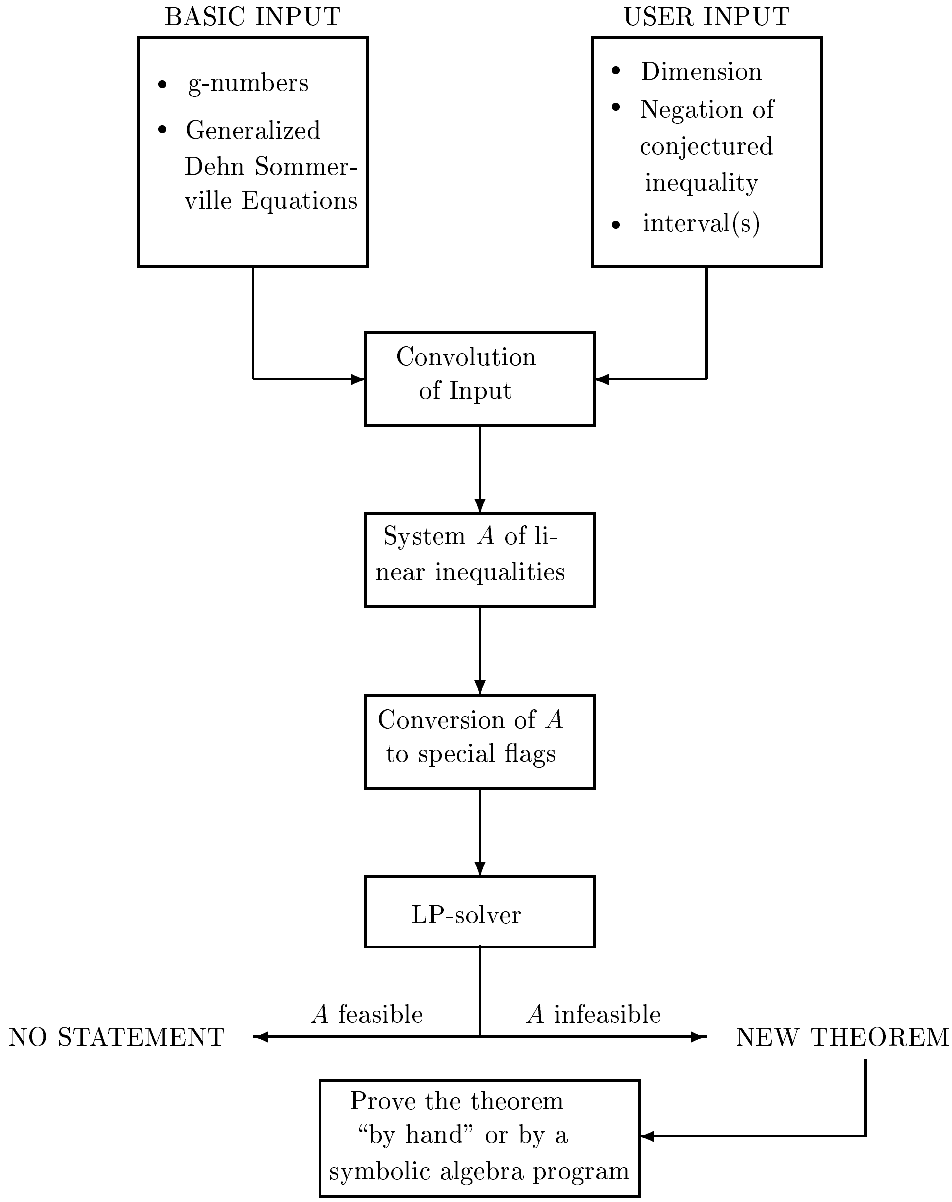

Figure 1: Automated theorem proving by FLAGTOOL

$$
\begin{aligned}
[1] \quad g_0^1 * g_1^2 * g_0^0 * g_1^2 * g_0^0 &= 6f_{0246} - 18f_{025} + 6f_{135} + 36f_{15} - 18f_{146} - 3f_{0257} + f_{1357} \\
&\quad +6f_{157} \\
[2] \quad g_1^2 * g_1^2 * g_1^2 * g_0^0 &= -54f_{25} - 9f_{257} - 54f_{36} + 27f_{036} + 54f_{26} - 18f_{035} + 36f_{35} \\
&\quad +9f_{246} - 3f_{0357} + 6f_{357} + 18f_{025} + 3f_{0257} - 9f_{136} \\
&\quad -18f_{026} + 6f_{135} - 3f_{0246} + f_{1357} \\
[3] \quad g_0^0 * g_2^4 * g_1^2 * g_0^0 &= -60f_{05} - 10f_{057} - 60f_{16} - 30f_{036} + 60f_{06} + 48f_{15} + 8f_{157} \\
&\quad -12f_{135} - 2f_{1357} + 18f_{035} + 3f_{0357} + 12f_{046} - 6f_{146} \\
&\quad -6f_{025} - f_{0257} + 18f_{136} + 12f_{026} \\
[4] \quad g_0^0 * g_1^2 * g_1^4 * g_0^0 &= -6f_{146} + 6f_{046} + 6f_{147} - 6f_{047} + 30f_{03} + 9f_{035} - 15f_{036} \\
&\quad +15f_{037} + 30f_{14} - 30f_{04} + f_{0246} - f_{0247} - 20f_{13} - 5f_{024} \\
&\quad -6f_{135} + 10f_{136} - 10f_{137} \\
[5] \quad g_0^0 * g_1^2 * \overline{g_1}^4 * g_0^0 &= 30f_{03} + 15f_{035} - 15f_{036} + 9f_{037} + 30f_{14} - 30f_{04} - 20f_{13} \\
&\quad -5f_{024} - 10f_{135} + 10f_{136} - 6f_{137} \\
[6] \quad g_0^3 * g_2^4 * g_0^0 &= 3f_{246} - 3f_{146} + 3f_{046} - f_{247} + f_{147} - f_{047} + 20f_3 + 4f_{35} \\
&\quad -10f_{36} + 4f_{37} - 10f_{24} + 10f_{14} - 10f_{04} \\
[7] \quad (f_0 - 78) * g_2^4 * g_0^0 &= 3f_{0246} - f_{0247} + 20f_{03} + 4f_{035} - 10f_{036} + 4f_{037} - 10f_{024} \\
&\quad -234f_{246} + 228f_{146} - 228f_{046} + 78f_{247} - 76f_{147} + 76f_{047} \\
&\quad -1560f_3 - 312f_{35} + 780f_{36} - 312f_{37} + 780f_{24} - 760f_{14} \\
&\quad +760f_{04} \\
[8] \quad (f_2 - 78) * g_1^4 * g_0^0 &= -10f_{13} + 10f_{03} - 3f_{135} + 3f_{035} + 5f_{136} - 5f_{036} - 5f_{137} \\
&\quad +5f_{037} + 76f_{246} - 78f_{146} + 78f_{046} - 76f_{247} + 78f_{147} \\
&\quad -78f_{047} + 760f_3 + 228f_{35} - 380f_{36} + 380f_{37} - 380f_{24} \\
&\quad +390f_{14} - 390f_{04} \\
[9] \quad (f_2 - 78) * \overline{g_1}^4 * g_0^0 &= -10f_{13} + 10f_{03} - 5f_{135} + 5f_{035} + 5f_{136} - 5f_{036} - 3f_{137} \\
&\quad +3f_{037} + 760f_3 + 380f_{35} - 380f_{36} + 228f_{37} - 380f_{24} \\
&\quad +390f_{14} - 390f_{04}
\end{aligned}
$$

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\begin{array}{llcl}
[10] & (f_2-78)*g_2^4*g_0^0 & = & 20f_{13}-20f_{03}+4f_{135}-4f_{035}-10f_{136}+10f_{036}+4f_{137} \\
 & & & -4f_{037}-228f_{246}+234f_{146}-234f_{046}+76f_{247} \\
 & & & -78f_{147}+78f_{047}-1520f_3-304f_{35}+760f_{36}-304f_{37} \\
 & & & +760f_{24}-780f_{14}+780f_{04} \\
[11] & g_0^0*g_1^7*g_0^0 & = & 2f_{13}-2f_{14}+2f_{15}-2f_{16}+2f_{17}+6f_{02}-8f_{03}+8f_{04} \\
 & & & -8f_{05}+8f_{06}-8f_{07}-12f_1 \\
[12] & g_0^0*\overline{g_1}^7*g_0^0 & = & 8f_{02}-8f_{03}+8f_{04}-8f_{05}+8f_{06}-6f_{07}-16f_1 \\
[13] & \overline{g_1}^8*g_0^0 & = & -18+9f_0-9f_1+9f_2-9f_3+9f_4-9f_5+9f_6-7f_7 \\
[14] & \overline{g_2}^8*g_0^0 & = & 72-36f_0+36f_1-36f_2+36f_3-36f_4-36f_6+30f_5 \\
 & & & +3f_{46}-3f_{36}+3f_{26}-3f_{16}+3f_{06}-f_{47}+f_{37} \\
 & & & -f_{27}+f_{17}-f_{07}+22f_7 \\
[15] & \overline{g_3}^8*g_0^0 & = & -168+84f_0-84f_1+84f_2+84f_4+84f_6+7f_{27}-7f_{17} \\
 & & & +7f_{07}-42f_7-f_{047}+f_{147}-f_{247}+3f_{046}-3f_{146} \\
 & & & +3f_{246}-64f_3-3f_{37}-10f_{24}+10f_{14}-10f_{04} \\
 & & & +5f_{47}+4f_{25}-4f_{15}+4f_{05}-54f_5 \\
 & & & -15f_{46}+9f_{36}-19f_{26}+19f_{16}-19f_{06} \\
[16] & g_0^0*g_1^2*g_0^0*g_1^2*g_0^1 & = & 4f_{1357}-6f_{0357}+18f_{047}+3f_{0247}-18f_{147} \\
[17] & g_0^1*g_1^2*g_1^2*g_0^1 & = & -6f_{157}+3f_{0257}-f_{1357}-3f_{0247}+9f_{147} \\
[18] & g_0^4*g_1^2*g_0^1 & = & 2f_{57}-f_{057}+f_{157}-f_{257}+f_{357}-3f_{47} \\
[19] & \overline{g_1}^4*g_1^2*g_0^1 & = & -3f_{047}+3f_{147}-3f_{247}-10f_{57}+5f_{057}-5f_{157} \\
 & & & +5f_{257}-3f_{357}+15f_{47} \\
[20] & g_2^4*g_1^2*g_0^1 & = & 3f_{0257}-f_{0357}+20f_{57}-10f_{057}+4f_{157}-10f_{257}+4f_{357} \\
 & & & -30f_{47}+12f_{047}-3f_{0247}+9f_{247}-3f_{147} \\
[21] & g_0^2*g_1^4*g_0^1 & = & 2f_{37}-f_{037}+f_{137}-5f_{27} \\
[22] & g_1^2*g_1^4*g_0^1 & = & -6f_{37}+3f_{037}-f_{137}+15f_{27}-5f_{027} \\
[23] & g_1^2*\overline{g_1}^4*g_0^1 & = & -f_{0247}+f_{0257}+3f_{247}-3f_{257}-f_{137}+3f_{037}-6f_{37} \\
 & & & +15f_{27}-5f_{027} \\
[24] & g_1^2*g_2^4*g_0^1 & = & -6f_{357}+3f_{0357}-f_{1357}+24f_{37}-12f_{037}+4f_{137}-30f_{27} \\
 & & & +9f_{257}-3f_{247}+10f_{027}-3f_{0257}+f_{0247} \\
[25] & g_0^0*g_3^6*g_0^1 & = & 2f_{147}-8f_{157}+2f_{1357}-8f_{137}+30f_{17}-35f_{07}-5f_{027} \\
 & & & +f_{0257}-3f_{0357}+10f_{057}-4f_{047}+13f_{037} \\
[26] & \overline{g_2}^7*g_0^1 & = & 7f_{07}-7f_{17}+7f_{27}-7f_{37}+f_{357}-f_{257}+f_{157}-f_{057} \\
 & & & -4f_{57}+4f_{47}+14f_7 \\
[27] & g_1^2*g_1^2*g_0^0*g_1^2 & = & -54f_{26}-27f_{036}+54f_{36}+18f_{257}-9f_{246}+6f_{0357} \\
 & & & -12f_{357}+18f_{026}+9f_{136}-6f_{0257}+3f_{0246}-2f_{1357}
\end{array}
$$

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\begin{array}{llcl}
[28] & g_0^0 * g_2^4 * g_0^0 * g_1^2 & = & -60f_{06} + 30f_{036} + 60f_{16} + 20f_{057} - 16f_{157} + 4f_{1357} \\
 & & & +6f_{146} - 12f_{046} - 6f_{0357} - 12f_{026} - 18f_{136} + 2f_{0257} \\
[29] & g_0^1 * g_1^2 * g_0^1 * g_1^2 & = & 3f_{0257} - 6f_{157} - 3f_{0246} - f_{1357} + 9f_{146} \\
[30] & \overline{g_1}^4 * g_0^1 * g_1^2 & = & -3f_{046} + 3f_{146} - 3f_{246} - 3f_{357} + 5f_{257} - 5f_{157} \\
 & & & +5f_{057} - 10f_{57} + 15f_{46} \\
[31] & g_1^2 * g_0^0 * g_1^2 * g_1^2 & = & 18f_{246} - 54f_{36} + 27f_{036} - 18f_{037} + 36f_{37} - 18f_{247} - 3f_{0357} \\
 & & & +6f_{357} - 6f_{0246} - 9f_{136} + 6f_{137} + 6f_{0247} + f_{1357} \\
[32] & g_0^0 * g_1^2 * g_1^2 * g_1^2 & = & 18f_{046} - 18f_{146} + 18f_{037} + 3f_{0357} + 18f_{147} - 18f_{047} \\
 & & & +3f_{0246} + 18f_{136} - 12f_{137} - 3f_{0247} - 2f_{1357} - 27f_{036} \\
[33] & g_0^3 * g_1^2 * g_0^2 & = & f_{046} - f_{146} + f_{246} - 3f_{36} \\
[34] & (f_2 - 78) * g_0^2 * g_1^2 & = & 3f_{036} - 3f_{136} + 2f_{137} - 2f_{037} + f_{1357} - f_{0357} - 152f_{37} \\
 & & & -76f_{357} + 76f_{247} - 78f_{147} + 78f_{047} + 228f_{36} \\
[35] & g_0^1 * g_2^4 * g_0^2 & = & f_{0246} - 4f_{026} + 10f_{16} - 3f_{146} + f_{136} \\
[36] & g_0^1 * g_2^4 * g_1^2 & = & -3f_{0246} + 12f_{026} + 3f_{0247} - f_{0257} + 20f_{17} + 4f_{137} - 10f_{147} \\
 & & & +4f_{157} - 10f_{027} - 30f_{16} + 9f_{146} - 3f_{136} \\
[37] & g_0^6 * g_1^2 & = & 2f_7 - f_{07} + f_{17} - f_{27} + f_{37} - f_{47} + f_{57} - 3f_6 \\
[38] & \overline{g_1}^6 * g_1^2 & = & -14f_7 + 7f_{07} - 7f_{17} + 7f_{27} - 7f_{37} + 7f_{47} - 3f_{06} + 3f_{16} \\
 & & & -3f_{26} + 3f_{36} - 3f_{46} - 5f_{57} + 21f_6 \\
[39] & g_0^0 * g_2^4 * g_0^3 & = & 2f_{135} - 8f_{15} + 10f_{05} - 3f_{035} + f_{025} \\
[40] & g_0^0 * g_1^2 * g_0^0 * g_2^4 & = & -60f_{04} + 60f_{14} + 24f_{035} - 6f_{0357} + 18f_{047} - 18f_{147} - 6f_{046} \\
 & & & +6f_{146} - 10f_{024} - 16f_{135} + 4f_{1357} + 3f_{0247} - f_{0246} \\
[41] & (f_2 - 78) * g_0^0 * g_2^4 & = & -8f_{135} + 8f_{035} + 2f_{1357} - 2f_{0357} - 780f_{04} + 780f_{14} \\
 & & & -760f_{24} + 608f_{35} - 152f_{357} + 234f_{047} - 234f_{147} \\
 & & & +228f_{247} - 78f_{046} + 78f_{146} - 76f_{246} \\
[42] & g_0^1 * g_1^2 * g_2^4 & = & 10f_{024} - 3f_{0247} + f_{0246} + 24f_{15} + 4f_{135} - 12f_{025} - 6f_{157} \\
 & & & -f_{1357} + 3f_{0257} - 3f_{146} + 9f_{147} - 30f_{14} \\
[43] & g_0^4 * g_2^4 & = & -8f_5 + 4f_{05} - 4f_{15} + 4f_{25} - 4f_{35} + 2f_{57} - f_{057} + f_{157} \\
 & & & -f_{257} + f_{357} + f_{46} - 3f_{47} + 10f_4 \\
[44] & \overline{g_1}^4 * g_0^4 & = & f_{04} - f_{14} + f_{24} - 5f_4 \\
[45] & (f_2 - 78) * g_0^5 & = & -f_{03} + f_{13} - 76f_3 \\
[46] & g_1^2 * g_3^6 & = & -90f_3 + 45f_{03} + 24f_{35} - 12f_{035} - 6f_{357} + 3f_{0357} + 24f_{37} \\
 & & & -12f_{037} - 6f_{36} + 3f_{036} - 15f_{13} + 4f_{135} - f_{1357} + 4f_{137} \\
 & & & -f_{136} + 13f_{025} - 4f_{026} + 10f_{027} - 3f_{0257} + f_{0247} - 5f_{024} \\
 & & & -35f_{02} - 39f_{25} + 12f_{26} - 30f_{27} + 9f_{257} - 3f_{247} + 15f_{24} + 105f_2
\end{array}
$$

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\begin{array}{llcl}
[47] & g_0^1 * g_1^7 & = & f_{02} - 8f_1 \\
[48] & g_0^1 * \overline{g_1}^7 & = & f_{13} - f_{14} + f_{15} - f_{16} + f_{17} - f_{02} - 6f_1 \\
[49] & g_0^0 * \overline{g_1}^8 & = & -f_{02} + f_{03} - f_{04} + f_{05} - f_{06} + f_{07} + 2f_1 - 9f_0 \\
[50] & g_0^0 * g_2^8 & = & -16f_1 + 2f_{13} + f_{02} - 3f_{03} + 36f_0 \\
[51] & g_0^0 * \overline{g_2}^8 & = & 8f_{02} - 8f_{03} + 8f_{04} - 8f_{05} - 16f_1 + 2f_{17} + f_{057} - f_{047} \\
 & & & +f_{037} - f_{027} - 7f_{07} + 5f_{06} + 36f_0 \\
[52] & g_0^0 * g_4^8 & = & -112f_1 + 30f_{13} - 8f_{135} - 10f_{14} + 2f_{147} - 8f_{157} + 2f_{1357} \\
 & & & -8f_{137} + 30f_{17} + 2f_{136} \\
 & & & -10f_{16} + 34f_{15} + 21f_{04} - 45f_{05} + 15f_{06} - 3f_{036} + f_{026} \\
 & & & -35f_{07} - 5f_{027} + f_{0257} - 3f_{0357} + 10f_{057} - 4f_{047} \\
 & & & +13f_{037} - 51f_{03} + 12f_{035} - 4f_{025} + 21f_{02} + 126f_0 \\
[53] & g_1^9 & = & -10 + f_0
\end{array}
$$

The proof was obtained as follows. FLAGTOOL creates 227 linear inequalities which contain the 53 inequalities above. The infeasibility of the entire list of inequalities and the creation of the smaller list of 53 inequalities which are already infeasible was first carried on using phase I of the LP-solver CPLEX. Since it was (at least theoretically) possible that the infeasibility found by CPLEX is an artifact of numerical problems we proved the infeasibility using the symbolic mathematical program MAPLE V.

Note that we need the nonnegativity of the forms g_3^d and it turns out that using only the nonnegativity of g_i^d, $i \leq 2$ is not sufficient to prove infeasibility. Thus, Theorem 1 is still unproved for general (non-rational) polytopes.

It seems impossible to prove the infeasibility "by hand". The details of this proof (which for a computer generated proof is quite short) do not seem to contribute much to our (human) insight for understanding why the theorem is true.

Proof of Theorem 2: We added the inequalities $f_0^3 - 5 \geq 0$ and $f_0^3 - 5 \geq 3$ (for all intervals of rank 4) and all inequalities generated by convolutions. The theorem follows from the infeasibility of a system of 50 linear inequalities which is contained in the 243 inequalities produced by FLAGTOOL. This time, we use only the nonnegativity of g_i^d for $i \leq 2$. The infeasibility was again proved by using MAPLE V. We will skip the details.

Remark: It is possible that the statements of Theorem 1 and 2 remain true for arbitrary Eulerian *lattices* ([2, 17]) for some sufficiently large dimension. The same comment applies to all the conjectures mentioned in Section 1. For Eulerian lattices it is no longer true that the g_i^d are nonnegative for $i \geq 2$ but the nonnegativity of the g_1^d may suffice.

Proof of Theorem 3: We added the inequalities $f_0^3 - 9 \geq 0$ and $f_0^3 - 9 \geq 3$ to the other inequalities for 5-polytopes and used convolution to obtain a system of 14 inequalities which turned out to be infeasible. This time, we used only the nonnegativity of g_1^d and hence the proof applies to arbitrary Eulerian lattices.

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