19 POLYTOPE SKELETONS AND PATHS

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INTRODUCTION

The k-dimensional ***skeleton*** of a d-polytope P is the set of all faces of the polytope of dimension at most k. The 1-skeleton of P is called the ***graph*** of P and denoted by $G(P)$. $G(P)$ can be regarded as an abstract graph whose vertices are the vertices of P, with two vertices adjacent if they form the endpoints of an edge of P.

In this chapter, we will describe results and problems concerning graphs and skeletons of polytopes. In Section 19.1 we briefly describe the situation for 3-polytopes. In Section 19.2 we consider general properties of polytopal graphs—subgraphs and induced subgraphs, connectivity and separation, expansion, and other properties. In Section 19.3 we discuss problems related to diameters of polytopal graphs in connection with the simplex algorithm and the Hirsch conjecture. The short Section 19.4 is devoted to polytopal digraphs. Section 19.5 is devoted to skeletons of polytopes, connectivity, collapsibility and shellability, empty faces and polytopes with "few vertices," and the reconstruction of polytopes from their low-dimensional skeletons; finally we consider what can be said about the collections of all k-faces of a d-polytope, first for $k = d - 1$ and then when k is fixed and d is large compared to k.

19.1 THREE-DIMENSIONAL POLYTOPES

GLOSSARY

Convex polytopes and their *faces* (and, in particular their *vertices*, *edges*, and *facets*) are defined in Chapter 15 of this Handbook.

A graph is ***d-polytopal*** if it is the graph of some d-polytope.

The following standard graph-theoretic concepts are used: *subgraphs*, *induced subgraphs*, the *complete graph* K_n on n vertices, *cycles*, *trees*, a *spanning tree* of a graph, *valence* (or *degree*) of a vertex in a graph, *planar* graphs, *d-connected* graphs, *coloring* of a graph, *subdivision* of a graph, and *Hamiltonian* graphs.

We briefly discuss results on 3-polytopes. Some of the following theorems are the starting points of much research, sometimes of an entire theory. Only in a few cases are there high-dimensional analogues, and this remains an interesting goal for further research.

THEOREM 19.1.1 *Whitney [104]* (1932)

Let G be the graph of a 3-polytope P. Then the graphs of faces of P are precisely the induced cycles in G that do not separate G.

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THEOREM 19.1.2 *Steinitz [96]* (1916)

A graph G is a graph of a 3-polytope if and only if G is planar and 3-connected.

Steinitz's theorem is the first of several theorems that describe the tame behavior of 3-polytopes. These theorems fail already in dimension four; see Chapter 15.

The theory of planar graphs is a wide and rich theory. Let us quote here the fundamental theorem of Kuratowski.

THEOREM 19.1.3 *Kuratowski ([71, 98])*

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$.

THEOREM 19.1.4 *Lipton and Tarjan [78]* (1979), *in a stronger form given by Miller [83]* (1986)

The graph of every 3-polytope with n vertices can be separated, by $2\sqrt{2n}$ vertices forming a circuit in the graph, into connected components of size at most $2n/3$.

It is worth mentioning that the Koebe circle packing theorem gives a new approach to both the Steinitz and Lipton-Tarjan theorems. (See [108, 88]).

Euler's formula $V - E + F = 2$ has many applications concerning graphs of 3-polytopes; in higher dimensions, our knowledge of face numbers of polytopes (see Chapter 17) applies to the study of their graphs and skeletons. Simple applications of Euler's theorem are:

THEOREM 19.1.5

Every 3-polytopal graph has a vertex of valence at most 5. (Equivalently, every 3-polytope has a face with at most five sides.)

THEOREM 19.1.6

Every 3-polytope has either a trivalent vertex or a triangular face.

A deeper application of Euler's theorem is:

THEOREM 19.1.7 *Kotzig [70]* (1955)

Every 3-polytope has two adjacent vertices the sum of whose valences is at most 13.

For a simple 3-polytope P, let $p_k = p_k(P)$ be the number of k-sized faces of P.

THEOREM 19.1.8 *Eberhard [29]* (1891)

For every finite sequence (p_k) of nonnegative integers with $\sum_{k\geq 3}(6-k)p_k = 12$, there exists a simple 3-polytope P with $p_k(P) = p_k$ for every $k \neq 6$.

Eberhard's theorem is the starting point of a large number of results and problems, see. e.g., [51, 46, 40]. While no high-dimensional direct analogues are known or even conjectured, the results and problems on facet-forming polytopes and nonfacets mentioned below seem related.

THEOREM 19.1.9 *Motzkin [85]* (1964)

The graph of a simple 3-polytope whose facets have 0 (mod 3) vertices has, all together, an even number of edges.

THEOREM 19.1.10 *Barnette [10]* (1966)

Every 3-polytopal graph contains a spanning tree of maximal valence 3.

We will now describe some results and a conjecture on colorability and Hamiltonian circuits.

THEOREM 19.1.11 *Four Color Theorem: Appel and Haken [4, 5, 92]* (1977)

The graph of every 3-polytope is 4-colorable.

THEOREM 19.1.12 *Tutte [99]* (1956)

4-connected planar graphs are Hamiltonian.

Tait conjectured in 1880, and Tutte disproved in 1946, that the graph of every simple 3-polytope is Hamiltonian. This started a rich theory of trivalent planar graphs without large paths.

CONJECTURE 19.1.13 *Barnette*

Every graph of a simple 3-polytope whose facets have an even number of vertices is Hamiltonian.

Finally, there are several exact and asymptotic formulas for the numbers of distinct graphs of 3-polytopes. A remarkable enumeration theory was developed by Tutte and was further developed by several authors. We will quote one result.

THEOREM 19.1.14 *Tutte [100]* (1962)

The number of rooted simplicial 3-polytopes with v vertices is

$$\frac{2(4v-11)!}{(3v-7)!(v-2)!}.$$

Tutte's theory provides also efficient algorithms to generate random planar graphs of various types.

PROBLEM 19.1.15

How does a random 3-polytopal graph look like?

Motivation to study this problem (and high-dimensional extensions) comes also from physics (specifically, "quantum gravity"). See [1, 3, 23]. One surprising property of random planar maps of various kinds is that the expected number of vertices of distance at most r from a given vertex behaves like r^4. (Compared to r^2 for the planar grid.)

19.2 GRAPHS OF d-POLYTOPES—GENERALITIES

GLOSSARY

For a graph G, TG denotes any ***subdivision*** of G, i.e., any graph obtained from G by replacing the edges of G by paths with disjoint interiors.

A d-polytope P is ***simplicial*** if all its proper faces are simplices. P is ***simple*** if every vertex belongs to d edges or, equivalently, if the polar of P is simplicial. P is ***cubical*** if all its proper faces are cubes.

A simplicial polytope P is ***stacked*** if it is obtained by the repeated operation of gluing a simplex along a facet.

For the definition of the *cyclic polytope* $C(d,n)$, see Chapter 15.

For two graphs G and H (considered as having disjoint sets V and V' of vertices), $G+H$ denotes the graph on $V \cup V'$ that contains all edges of G and H together with all edges of the form $\{v,v'\}$ for $v \in V$ and $v' \in V'$.

A graph G is ***d-connected*** if G remains connected after the deletion of any set of at most $d-1$ vertices.

An ***empty simplex*** of a polytope P is a set S of vertices such that S does not form a face but every proper subset of S forms a face.

A graph G whose vertices are embedded in $\mathbb{R}^d$ is ***rigid*** if every small perturbation of the vertices of G that does not change the distance of adjacent vertices in G is induced by an affine rigid motion of $\mathbb{R}^d$. G is ***generically d-rigid*** if it is rigid with respect to "almost all" embeddings of its vertices into $\mathbb{R}^d$. (Generic rigidity is thus a graph theoretic property, but no description of it in pure combinatorial terms is known for $d > 2$; cf. Chapter 59.)

A set A of vertices of a graph G is ***totally separated*** by a set B of vertices, if A and B are disjoint and every path between two distinct vertices in A meets B.

A graph G is an ***ϵ-expander*** if, for every set A of at most half the vertices of G, there are at least $\epsilon \cdot |A|$ vertices not in A that are adjacent to vertices in A.

Neighborly polytopes and $(0,1)$-*polytopes* are defined in Chapter 15.

The *polar dual* P^Δ of a polytope P is defined in Chapter 15.

SUBGRAPHS AND INDUCED SUBGRAPHS

THEOREM 19.2.1 *Grünbaum [35]* (1965)

Every d-polytopal graph contains a TK_{d+1}.

THEOREM 19.2.2 *Kalai [53]*(1987)

The graph of a simplicial d-polytope P contains a TK_{d+2} if and only if P is not stacked.

One important difference between the situation for $d = 3$ and for $d > 3$ is that K_n, for every $n > 4$, is the graph of a 4-dimensional polytope (e.g., a cyclic polytope). Simple manipulations on the cyclic 4-polytope with n vertices show:

PROPOSITION 19.2.3 *Perles (unpublished)*

(i) *Every graph G is a spanning subgraph of the graph of a* 4*-polytope.*

(ii) *For every graph G, $G + K_n$ is a d-polytopal graph for some n and some d.*

This proposition extends easily to higher-dimensional skeletons in place of graphs. It is not known what the minimal dimension is for which $G + K_n$ is d-polytopal, nor even whether $G + K_n$ (for some $n = n(G)$) can be realized in some bounded dimension uniformly for all graphs G.

CONNECTIVITY AND SEPARATION

THEOREM 19.2.4 *Balinski [8]*(1961)

The graph of a d-polytope is d-connected.

A set S of d vertices that separates P must form an empty simplex; in this case, P can be obtained by gluing two polytopes along a simplex facet of each.

THEOREM 19.2.5 *Larman and Mani[73]* (1970)

Let G be the graph of a d-polytope. Let $e = \lfloor (d+1)/3 \rfloor$. Then for every two disjoint sequences $(v_1, v_2, \ldots, v_e)$ and $(w_1, w_2, \ldots w_e)$ of vertices of G, there are e vertex-disjoint paths connecting v_i to w_i, $i = 1, 2, \ldots, e$.

PROBLEM 19.2.6 *Larman*

Is the last theorem true for $e = \lfloor d/2 \rfloor$?

THEOREM 19.2.7 *Cauchy, Dehn, Alexandrov, Whiteley, ...*

(i) *For, $d = 3$: Cauchy's theorem. If P is a simplicial d-polytope, $d \geq 3$, then $G(P)$ (with its embedding in $\mathbb{R}^d$) is rigid.*

(ii) *Whiteley's theorem. For a general d-polytope P, let G' be a graph (embedded in $\mathbb{R}^d$) obtained from $G(P)$ by triangulating the 2-faces of P without introducing new vertices. Then G' is rigid.*

COROLLARY 19.2.8

For a simplicial d-polytope P, $G(P)$ is generically d-rigid. For a general d-polytope P and a graph G' (considered as an abstract graph) as in the previous theorem, G' is generically d-rigid.

The main combinatorial application of the above theorem is the Lower Bound Theorem (see Chapter 17) and its extension to general polytopes.

Note that Corollary 19.2.8 can be regarded also as a strong form of Balinski's theorem. It is well known and easy to prove that a generic d-rigid graph is d-connected. Therefore, for simplicial (or even 2-simplicial) polytopes Corollary 19.2.8 implies directly that $G(P)$ is d-connected.

For general polytopes we can derive Balinski's theorem as follows. Suppose to the contrary that the graph G of a general d-polytope P is not d-connected and therefore its vertices can be separated into two parts (say, red vertices and blue vertices) by deleting a set A of $d-1$ vertices. It is easy to see that every 2-face of P can be triangulated without introducing a blue-red edge. Therefore, the resulting triangulation is not $(d-1)$-connected and hence it is not generically d-rigid. This contradicts the assertion of Corollary 19.2.8.

Let $\mu(n, d) = f_{d-1}(C(d, n))$ be the number of facets of a cyclic d-polytope with n vertices, which, by the Upper Bound Theorem, is the maximal number of facets possible for a d-polytope with n vertices.

THEOREM 19.2.9 *Klee [63]* (1964)

The number of vertices of a d-polytope that can be totally separated by n vertices is at most $\mu(n, d)$.

Klee also showed by considering cyclic polytopes with simplices stacked to each of their facets that this bound is sharp . It follows that there are graphs of simplicial d-polytopes which are not graphs of $(d-1)$-polytopes. (After realizing that the complete graphs are 4-polytopal a naive thought would be that every d-polytopal graph is 4-polytopal.)

EXPANSION

Expansion properties for the graph of the d-dimensional cube are known and important in various areas of combinatorics. By direct combinatorial methods, one can obtain expansion properties of duals to cyclic polytopes. There are a few positive results and several interesting conjectures on expansion properties of graphs of large families of polytopes.

THEOREM 19.2.10 *Kalai [56]*(1992)

Graphs of duals to neighborly d-polytopes with n facets are ϵ-expanders for $\epsilon = O(n^{-4})$.

This result implies that the diameter of graphs of duals to neighborly d-polytopes with n facets is $O(d \cdot n^4 \cdot \log n)$.

CONJECTURE 19.2.11 *Mihail and Vazirani*

Graphs of $(0,1)$-polytopes P have the following expansion property: For every set A of at most half the vertices of P, the number of edges joining vertices in A to vertices not in A is at least $|A|$.

The Mihail and Vazirani conjecture is from Feder and Mihail [30]; Recent reference: Kaibel [52]

It is also conjectured that graphs of polytopes cannot have very good expansion properties:

CONJECTURE 19.2.12 *Graphs of polytopes are not very good expanders, [56]*

Let d be fixed. The graph of every simple d-polytope with n vertices can be separated into two parts, each having at least $n/3$ vertices, by removing $O(n^{1-1/(d-1)})$ vertices.

It is known that there are dual graphs to triangulations of S^3 which cannot be separated even by $O(n/\log n)$ vertices [84]. Graphs of dual to cyclic $2k$-polytopes with n vertices when n is large looks somewhat like graphs of grids in $\mathbf{Z}^k$ and, in particular, we cannot find for them a separator of size $o(n^{1-1/k})$.

CONJECTURE 19.2.13 *Expansion properties of random polytopes, [56]*

A random simple d-polytope with n facets is an $O(1/(n-d))$-expander.

This conjecture is vaguely stated since there are various models for random polytopes. There are models based on geometric notion on randomness. For example consider polytopes (containing the origin) which are determined by n random hyperplanes that are tangent to the unit sphere. There is much recent interest in random gaussian perturbations of a fixed simple polytope [95]. We can also consider a random combinatorial type.

CONJECTURE 19.2.14 *There are only a "few" graphs of polytopes*

The number of distinct (isomorphism types) of graphs of simple d-polytopes with n vertices is at most C_d^n, where C_d is a constant depending on d.

It is even possible that the same constant applies for all dimensions and that the conjecture holds even for graphs of general polytopes. This conjecture is of interest also for dual graphs of triangulations of spheres. Conjecture 19.2.12 (and even a much weaker seperation property) would imply Conjecture 19.2.14.

OTHER PROPERTIES

CONJECTURE 19.2.15 *Barnette*

Every graph of a simple d-polytope, $d \geq 4$, is Hamiltonian.

THEOREM 19.2.16 (1978)

For a simple d-polytope P, $G(P)$ is 2-colorable if and only if $G(P^\Delta)$ is d-colorable.

This theorem was proved in an equivalent form for $d = 4$ by Goodman and Onishi [34]. (For $d = 3$ it is a classical theorem by Ore.) For the general case, see Joswig [48]. This theorem is related to seeking two-dimensional analogs of Hamiltonian cycles in skeletons of polytopes and manifolds, see [94].

19.3 DIAMETERS OF POLYTOPAL GRAPHS

GLOSSARY

A ***d-polyhedron*** is the intersection of a finite number of halfspaces in $\mathbb{R}^d$.

$\Delta(d,n)$ denotes the maximal diameter of the graphs of d-dimensional polyhedra P with n facets.

$\Delta_b(d,n)$ denotes the maximal diameter of the graphs of d-polytopes with n vertices.

Given a d-polyhedron P and a linear functional ϕ on $\mathbb{R}^d$, we denote by $G^{\rightarrow}(P)$ the directed graph obtained from $G(P)$ by directing an edge $\{v,u\}$ from v to u if $\phi(v) \leq \phi(u)$. $v \in P$ is a ***top vertex*** if ϕ attains its maximum value in P on v.

Let $H(d,n)$ be the maximum over all d-polyhedra with n facets and all linear functionals on $\mathbb{R}^d$ of the maximum length of a minimal monotone path from any vertex to a top vertex.

Let $M(d,n)$ be the maximal number of vertices in a monotone path over all d-polyhedra with n facets and all linear functionals on $\mathbb{R}^d$.

For the notions of *simplicial complex*, *polyhedral complex*, *pure simplicial complex*, and the *boundary complex* of a polytope, see Chapter 17.

Given a pure $(d-1)$-dimensional simplicial (or polyhedral) complex K, the ***dual graph*** $G^\Delta(K)$ of K is the graph whose vertices are the facets ($(d-1)$-faces) of K, with two facets F, F' adjacent if dim $(F \cap F') = d-2$.

A pure simplicial complex K is ***vertex-decomposable*** if there is a vertex v of K such that $\mathrm{lk}(v) = \{S \backslash \{v\} \mid S \in K, v \in S\}$ and $\mathrm{ast}(v) = \{S \mid S \in K, v \notin S\}$ are both vertex-decomposable. (The complex $K = \{\emptyset\}$ consisting of the empty face alone is vertex-decomposable.)

It is a long-outstanding open problem to determine the behavior of the function $\Delta(d,n)$. In 1957, Hirsch conjectured that $\Delta(d,n) \leq n-d$. Klee and Walkup [67] showed that the Hirsch conjecture is false for unbounded polyhedra. The Hirsch conjecture for bounded polyhedra is still open. The special case asserting that $\Delta_{\mathrm{b}}(d,2d) = d$ is called the ***d-step conjecture***, and it was shown by Klee and Walkup to imply that $\Delta_{\mathrm{b}}(d,n) \leq n-d$. Another equivalent formulation is that between any pair of vertices v and w of a polytope P there is a ***nonrevisiting path***, i.e., a path $v = v_1, v_2, ..., v_m = w$ such that for every facet F of P, if $v_i, v_j \in F$ for $i < j$ then $v_k \in F$ for every k, $i \leq k \leq j$.

THEOREM 19.3.1 *Klee and Walkup* (1967)

$$\Delta(d,n) \geq n - d + \min\{\lfloor d/4 \rfloor, \lfloor (n-d)/4 \rfloor\}.$$

THEOREM 19.3.2 *Holt and Klee [43, 44, 45]* (1998), *Fritzsche and Holt [33]*(2000)

For $n > d \geq 8$

$$\Delta_{\mathrm{b}}(d,n) \geq n - d.$$

THEOREM 19.3.3 *Barnette[13]* (1974)

$$\Delta(d,n) \leq \frac{2}{3} \cdot (n - d + 5/2) \cdot 2^{d-3}.$$

THEOREM 19.3.4 *Kalai and Kleitman [60]* (1992)

$$\Delta(d,n) \leq n \cdot \binom{\log n + d}{d} \leq n^{\log d + 1}.$$

The major open problem in this area is:

PROBLEM 19.3.5 *Is there a polynomial upper bound for $\Delta(d,n)$? Is there a linear upper bound for $\Delta(d,n)$?*

Some special classes of polytopes are known to satisfy the Hirsch bound or to have upper bounds for their diameters that are polynomial in d and n.

THEOREM 19.3.6 *Provan and Billera [91]* (1980)

Let G be the dual graph that corresponds to a vertex-decomposable $(d-1)$-dimensional simplicial complex with n vertices. Then the diameter of G is at most $n-d$.

It is known that this theorem does not imply the Hirsch conjecture (for polytopes) since there are simplicial polytopes whose boundary complexes are not vertex-decomposable. Yet, such examples are not so easy to come by.

THEOREM 19.3.7 *Naddef [86]* (1989)

The graph of every $(0,1)$ d-polytope has diameter at most d.

Balinski [9] proved the Hirsch bound for dual transportation polytopes, Dyer and Frieze [28] showed a polynomial upper bound for unimodular polyhedra, Kalai

[57] observed that if the ratio between the number of facets and the dimension is bounded above for the polytope and all its faces then the diameter is bounded above by a polynomial in the dimension, Kleinschmidt and Onn [68] proved extensions of Naddef's results to integral polytopes, and Deza and Onn [27] found upper bounds for the diameter in terms of lattice points in the polytope.

The value of $\Delta(d, n)$ is a lower bound for the number of iterations needed for Dantzig's simplex algorithm for linear programming with any pivot rule. However, it is still an open problem to find pivot rules where each pivot step can be computed with a polynomial number of arithmetic operations in d and n such that the number of pivot steps needed comes close to the upper bounds for $\Delta(d, n)$ given above. See Chapter 44.

The problem of routing in graphs of polytopes, namely finding a path between two vertices is an interesting computational problem.

PROBLEM 19.3.8 *Find an efficient routing algorithm for convex polytopes*

Using linear programming it is possible to find a path in a polytope P between two vertices that obeys the upper bounds given above such that the number of calls to the linear programming subroutine is roughly the number of edges of the path. Finding a routing algorithm for polytopes with a "small" number of arithmetic operations as a function of d and n is an interesting challenge. The subexponential simplex type algorithms (see Chapter 44) yield subexponential routing algorithm, but improvement for routing beyond what is known for linear programming is possible.

The upper bounds for $\Delta(d, n)$ mentioned above apply even to $H(d, n)$. Klee and Minty considered a certain geometric realization of the d-cube to show that

THEOREM 19.3.9 *Klee and Minty [66]* (1972)

$M(d, 2d) \geq 2^d$.

Recent far-reaching extensions of the Klee-Minty construction were found by Amenta and Ziegler[2] . It is not known for $d > 3$ and $n \geq d + 3$ what is the precise upper bound for $M(d, n)$ and does it coincide with the maximum number of vertices of a d-polytope with n facets given by the upper bound theorem (Chapter 17). See, Pfeifle [90].

19.4 POLYTOPAL DIGRAPHS

Given a d-polytope P and a linear objective function ϕ such that ϕ is not constant on edges, direct every edge of $G(P)$ towards the vertex with the higher value of the objective function. A directed graph obtained in this way is called a ***polytopal digraph***.

The following basic result is fundamental for the simplex algorithm and also has many applications for the combinatorial theory of polytopes.

THEOREM 19.4.1 *Folklore, see, e.g., [105]*

A polytopal digraph has one sink (and one source). Morover, every induced subgraphs on vertices of any face F of the polytope has one sink (and one source).

An acyclic orientation of $G(P)$ with the property that every face has a unique sink is called an ***abstract objective function***. Joswig, Kaibel and Körner [49] showed that an acyclic orientation for which every 2-dimensional face has a unique sink is already an abstract objective function.

The h-vector of a simplicial polytope P has a simple and important interpretation in terms of the directed graph that corresponds to the polar of P. The number $h_k(P)$ is the number of vertices v of P^Δ of outdegree k. (Recall that every vertex in a simple polytope has exactly d neighboring vertices.) Switching from ϕ to $-\phi$, one gets the Dehn-Sommerville relations $h_k = h_{d-k}$ (including the Euler relation for $k = 0$); see Chapter 17.

Studying polytopal digraphs and digraphs obtained by abstract objective functions is very interesting in the three dimensional case and in high dimensions.

THEOREM 19.4.2 *Mihalisin and Klee [82]*(2000)

Suppose that K is an orientation of a 3-polytopal graph G. Then the digraph K is 3-polytopal if and only if it is acyclic, has a unique source and a unique sink, and admits three independent monotone paths from the source to the sink.

Mihalisin and Klee write in their article "we hope that the present article will open the door to a broader study of polytopal digraphs".

19.5 SKELETONS OF POLYTOPES

GLOSSARY

A pure polyhedral complex K is ***strongly connected*** if its dual graph is connected.

A ***shelling order*** of the facets of a polyhedral $(d-1)$-dimensional sphere is an ordering of the set of facets $F_1, F_2, \ldots, F_n$ so that the simplicial complex K_i spanned by $F_1 \cup F_2 \cup \cdots \cup F_i$ is a simplicial ball for every $i < n$. A polyhedral complex is ***shellable*** if there exists a shelling order of its facets.

A simplicial polytope is ***extendably shellable*** if any way to start a shelling can be continued to a shelling.

An ***elementary collapse*** on a simplicial complex is the deletion of two faces F and G so that F is maximal and G is a codimension-1 face of F that is not included in any other maximal face. A polyhedral complex is ***collapsible*** if it can be reduced to the void complex by repeated applications of elementary collapses.

A d-dimensional polytope P is ***facet-forming*** if there is a $(d+1)$-dimensional polytope Q such that all facets of Q are combinatorially isomorphic to P. If no such Q exists, P is called a ***nonfacet***.

A ***rational polytope*** is a polytope whose vertices have rational coordinates. (Not every polytope is combinatorially isomorphic to a rational polytope; see Chapter 15.)

A d-polytope P is ***k-simplicial*** if all its faces of dimension at most k are simplices. P is ***k-simple*** if its polar dual P^Δ is k-simplicial.

Zonotopes are defined in Chapters 15 and 17.

Let K be a polyhedral complex. An ***empty simplex*** S of K is a minimal nonface of K, i.e., a subset S of the vertices of K with S itself not in K, but every proper subset of S in K.

Let K be a polyhedral complex and let U be a subset of its vertices. The ***induced subcomplex*** of K on U, denoted by $K[U]$, is the set of all faces in K whose vertices belong to U. An ***empty face*** of K is an induced polyhedral subcomplex of K that is homeomorphic to a polyhedral sphere. An empty 2-dimensional face is called an ***empty polygon***. An ***empty pyramid*** of K is an induced subcomplex of K that consists of all the proper faces of a pyramid over a face of K.

CONNECTIVITY AND SUBCOMPLEXES

THEOREM 19.5.1 *Grünbaum [35]* (1965)

The i-skeleton of every d-polytope contains a subdivision of $\mathrm{skel}_i(\Delta^d)$, the i-skeleton of a d-simplex.

THEOREM 19.5.2 *Folklore*

(i) *For $i > 0$, $\mathrm{skel}_i(P)$ is strongly connected.*

(ii) *For every face F, let $U_i(F)$ be the set of all i-faces of P containing F. Then if $i > \dim F$, $U_i(F)$ is strongly connected.*

Part (ii) follows at once from the fact that the faces of P containing F correspond to faces of the quotient polytope P/F. However, properties (i) and (ii) together are surprisingly strong, and all the known upper bounds for diameters of graphs of polytopes rely only on properties (i) and (ii) for the dual polytope.

THEOREM 19.5.3 *van Kampen and Flores [101, 32, 106]*(1935)

For $i \geq \lfloor d/2 \rfloor$, $\mathrm{skel}_i(\Delta^{d+1})$ is not embeddable in S^{d-1} (and hence not in the boundary complex of any d-polytope).

(This extends the fact that K_5 is not planar.)

CONJECTURE 19.5.4 *Lockeberg*

For every partition of $d = d_1 + d_2 + \cdots d_k$ and two vertices v and w of P, there are k disjoint paths between v and w such that the i th path is a path of d_i-faces in which any two consecutive faces have (d_i-1)-dimensional intersection.

SHELLABILITY AND COLLAPSIBILITY

THEOREM 19.5.5 *Bruggesser and Mani [22]*(1970)

Boundary complexes of polytopes are shellable.

The proof of Bruggesser and Mani is based on starting with a point near the center of a facet and moving from this point to infinity, and back from the other direction, keeping track of the order in which facets are seen. This proves a stronger form of shellability, in which each K_i is the complex spanned by all the facets that can be seen from a particular point in $\mathbb{R}^d$. It follows from shellability that

THEOREM 19.5.6

Polytopes are collapsible.

THEOREM 19.5.7 *Ziegler [107]* (1992)

There are d-polytopes, $d \geq 4$, whose boundary complexes are not extendably shellable.

THEOREM 19.5.8 *There are triangulations of the $(d-1)$-sphere which are not shellable.*

Lickorish [77] produced explicit examples of nonshellable triangulations of S^3. His result was that a triangulation containing a sufficiently complicated knotted triangle was not shellable. Hachimori and Ziegler [42] produced simple examples and showed that a triangulation containing any knotted triangle is not "constructible", constructibility being a strictly weaker notion than shellability. For more on shellability, see [25] [19].

FACET-FORMING POLYTOPES AND SMALL LOW-DIMENSIONAL FACES

THEOREM 19.5.9 *Perles and Shephard [89]*(1967)

Let P be a d-polytope such that the maximum number of k-faces of P on any $(d-2)$-sphere in the skeleton of P is at most $(d-1-k)/(d+1-k)f_k(P)$. Then P is a nonfacet.

An example of a nonfacet which is simple was found by Barnette [11]. . Some of the proofs of Perles and Shaphard uses metric properties of polytopes and for a few of the results alternatives proofs using shellability were found by Barnette [11].

THEOREM 19.5.10 *Schulte [93]* (1985)

The cuboctahedron and the icosidodecahedron are nonfacets.

PROBLEM 19.5.11

Is the icosahedron facet-forming?

For all other regular polytopes the situation is known. The simplices and cubes in any dimension and the 3-dimensional octahedron are facet-forming. All other regular polytopes with the exception of the icosahedron are known to be nonfacets.

It is very interesting to find what can be said on metrical properties of facets (or low dimensional faces) of a convex polytope.

THEOREM 19.5.12 *Barany (unpublished)*

There exist $\epsilon > 0$ such that every d-polytope, $d > 2$, has a facet F such that there is no ball B_1 of radius R and a ball B_2 of radius $(1+\epsilon)R$ such that $B_1 \subset F \subset B_2$.

The stronger statement where balls are replaced by elipses is open.

Next, we try to understand if it is possible for all the k-faces of a d-polytope to be isomorphic to a given polytope P. The following conjecture asserts that if d is large with respect to k, this can happen only if P is either a simplex or a cube.

CONJECTURE 19.5.13 *Kalai*

For every k there is a $d(k)$ such that every d-polytope with $d > d(k)$ has a k-face that is either a simplex or combinatorially isomorphic to a k-dimensional cube.

Recently, Julian Pfeifle showed based on the Wythoff construction (see Chapter 18) that $d(k) > (2k-1)(k-1)$, for $k \geq 3$.

For simple polytopes, it follows from the next theorem that if $d > ck^2$ then every d-polytope has a k-face F such that $f_r(F) \leq f_r(C_k)$. (Here, C_k denotes the k-dimensional cube.)

THEOREM 19.5.14 *Nikulin [87]* (1981)

The average number of r-dimensional faces of a k-dimensional face of a simple d-dimensional polytope is at most

$$\binom{d-r}{d-k} \cdot \left(\left(\binom{\lfloor d/2 \rfloor}{r} + \binom{\lfloor (d+1)/2 \rfloor}{r} \right) \Big/ \left(\binom{\lfloor d/2 \rfloor}{k} + \binom{\lfloor (d+1)/2 \rfloor}{k} \right) \right).$$

Nikulin's theorem appeared in his study of reflection groups in hyperbolic spaces. The existence of reflection groups of certain types implies some combinatorial conditions on their fundamental regions (which are polytopes) and Vinberg, Nikulin, Khovanski [102, 87, 62] and others showed that in high dimensions these combinatorial conditions lead to a contradiction. There are still many open problems in this direction. In particular, to narrow the gap between the dimensions above which those reflection groups cannot exists and the dimensions for which such groups can be constructed.

THEOREM 19.5.15 *Kalai [55]* (1989)

Every d-polytope for $d \geq 5$ has a 2-face with at most 4 vertices.

THEOREM 19.5.16 *Meisinger, Kalai and Kleinschmidt [80]* (2000)

Every rational d-polytope for $d \geq 9$ has a 3-face with at most 150 vertices.

The last two theorems and the next one are proved using the linear inequalities for flag numbers that are known via intersection homology of toric varieties; see Chapter 17. One can also study, in a similar fashion, quotients of polytopes.

CONJECTURE 19.5.17 *Perles*

For every k there is a $d'(k)$ such that every d-polytope with $d > d'(k)$ has a k-dimensional quotient that is a simplex.

As was mentioned in the first section, $d'(2) = 3$. The 24-cell, which is a regular 4-polytope all whose faces are octahedra, shows that $d'(3) > 4$.

THEOREM 19.5.18 *Meisinger, Kalai and Kleinschmidt [80]* (2000)

Every d-polytope with $d \geq 9$ has a 3-dimensional quotient that is a simplex.

PROBLEM 19.5.19

For which values of k and r are there d-polytopes other than the d-simplex that are *both k-simplicial and r-simple?*

It is known that this can happen only when $k+r \leq d$. There are infinite families of $(d-2)$-simplicial and 2-simple polytopes, and some examples of $(d-3)$-simplicial and 3-simple d-polytopes.

Concerning this problem Peter McMullen recently noted that the polytopes r_{st}, discussed in Coxeter's classic book on regular polytopes [24] in Sections 11.8, 11.x, are $(r+2)$-simplicial $d-r-2$-simple, where $d = r+s+t+1$. These Gosset-Elte polytopes arise by the so-called Wythoff construction from the finite reflection groups (see Chapter 18); we obtain a finite polytope whenever the reflection group generated by the Coxeter diagram with r, s, t nodes on the three arms are finite, that is, when

$$1/(r+1) + 1/(s+1) + 1/(t+1) > 1.$$

The largest exceptional example, 2_{41}, is related to the Weyl group E_8. The Gosset-Elte polytope 2_{41}, is a 4-simple 4-simplicial 8-polytope with 2160 vertices. Are there 5-simplicial 5-simple 10-polytopes?

THEOREM 19.5.20

For $d > 2$, there is no cubical d-polytope P whose dual is also cubical.

I am not aware of a reference for this result but it can easily be proved by showing a covering map from the standard cubical complex realizing R^{d-1} into K.

We talked about finding very special polytopes as "subobjects" (faces, quotients) of arbitrary polytopes. What about realizing arbitrary polytopes as "subobjects" of very special polytopes? There is an old conjecture that every polytope can be realized as a subpolytope (namely the convex hull of a subset of the vertices) of a stacked polytope. Perles and Sturmfels asked if every simlpicial d-polytope can be realized as the quotient of some neighborly even-dimensional polytope. (Recall that a $2m$-polytope is neighborly if every m vertices are the vertices of a $(m-1)$-dimensional face.) Kortenkamp [69] proved that this is the case for d-polytopes with at most $d+4$ vertices. For general polytopes "neighborly polytopes" should be replaced here by "weakly-neighborly" polytopes, introduced by Bayer [15], which are defined by the property that every set of k vertices is contained in a face of dimension at most $2k-1$. The only theorem of this flavour I am aware of is by Billera and Sarangarajan [17] who proved that every 0-1 polytope is a face of a travelling salesman polytope.

RECONSTRUCTION

THEOREM 19.5.21 *An extension of Whitney's theorem, [36]*

d-polytopes are determined by their $(d-2)$-skeletons.

THEOREM 19.5.22 *Perles* (1973) *(unpublished)*

Simplicial d-polytopes are determined by their $\lfloor d/2 \rfloor$-skeletons.

This follows from the following theorem (here, $\mathrm{ast}(F, P)$ is the complex formed by the faces of P that are disjoint to all vertices in F).

THEOREM 19.5.23 *Perles* (1973)

Let P be a simplicial d-polytope.

(i) *If F is a k-face of P, then* $\mathrm{skel}_{d-k-2}(\mathrm{ast}(F,P))$ *is contractible in* $\mathrm{skel}_{d-k-1}(\mathrm{ast}(F,P))$.

(ii) *If F is an empty k-simplex, then* $\mathrm{ast}(F,P)$ *is homotopically equivalent to* S^{d-k}*; hence,* $\mathrm{skel}_{d-k-2}(\mathrm{ast}(F,P))$ *is not contractible in* $\mathrm{skel}_{d-k-1}(\mathrm{ast}(F,P))$.

An extension of Perles' theorem for manifolds with a vanishing middle homology was proved by Dancis [26].

THEOREM 19.5.24 *Blind and Mani [21]* (1987)

Simple polytopes are determined by their graphs.

Blind and Mani described their theorem in a dual form and considered $(d-1)$-dimensional "puzzles" whose pieces are simplices and we wish to solve the puzzle based on the "local" information: which two simplices share a facet. Joswig extended their result to more genaral puzzles where the pieces are general $(d-1)$-dimensional polytopes and the way every two pieces which share a facet are connected is also prescribed. A simple proof is given in [54]. This proofs also shows that k-dimensional skeletons of simplicial polytopes are also determined by their "puzzle". Combined with Perles' theorem it follows that:

THEOREM 19.5.25 *Kalai and Perles* (1988)

Simplicial d-polytopes are determined by the incidence relations between i- and $(i+1)$-faces for every $i > \lfloor d/2 \rfloor$.

CONJECTURE 19.5.26 *Haase and Ziegler*

Let G be a graph of a simple 4-polytope. Let H be an induced, non-separating, 3-regular, 3-connected planar subgraph of G. Then H is the graph of a facet of P.

Haase and Ziegler [41] showed that this is not the case if H is not planar. Their proof touches on the issue of embedding knots in skelatons of 4-polytopes.

PROBLEM 19.5.27 *Are simplicial spheres determined by the incidence relation between facets and sub-facets?*

THEOREM 19.5.28 *Björner, Edelman, and Ziegler [20]* (1990)

Zonotopes are determined by their graphs.

THEOREM 19.5.29 *Babson, Finschi and Fukuda [7]* (2001)

Duals of cubical zonotopes are determined by their graphs.

In all instances of the above theorems except the single case of the Blind-Mani theorem, the proofs give reconstruction algorithms that are polynomial in the data. It is an open question if a polynomial algorithm exists to determine a simple polytope from its graph. A polynomial "certificate" for reconstruction was recently found by Joswig, Kaibel and Körner [49].

An interesting problem was whether there is an e-dimensional polytope other than the d-cube with the same graph as the d-cube?

THEOREM 19.5.30 *Joswig and Ziegler [50]* (2000)

For every $d \geq e \geq 4$ there is an e-dimensional cubical polytopes with 2^d vertices whose $\lfloor e/2 \rfloor - 1$-skeleton is combinatorially isomorphic to the $\lfloor e/2 \rfloor - 1$-skeleton of a d-dimensional cube.

Earlier Babson, Billera, and Chan [6] found such a construction for cubical spheres.

Another issue of reconstruction for polytopes that was studied extensively is the following: In which cases does the combinatorial structure of a polytope determine its geometric structure (up to projective transformations)? Such polytopes are called ***projectively unique***, and the major unsolved problem is:

PROBLEM 19.5.31

Are there only finitely many projectively unique polytopes in each dimension?

McMullen [79] constructed projectively unique d-polytopes with $3^{d/3}$ vertices.

EMPTY FACES AND POLYTOPES WITH FEW VERTICES

THEOREM 19.5.32 *Perles (unpublished manuscript)* (1970)

Let $f(d, k, b)$ be the number of combinatorial types of k-skeletons of d-polytopes with $d + b + 1$ vertices. Then, for fixed b and k, $f(d, k, b)$ is bounded.

This follows from

THEOREM 19.5.33 *Perles* (1970)

The number of empty i-pyramids for d-polytopes with $d + b$ vertices is bounded by a function of i and b.

For another proof of this theorem see [58].

For a d-polytope P, let $e_i(P)$ denote the number of empty i-simplices of P.

PROBLEM 19.5.34

Characterize the sequence of numbers $(e_1(P), e_2(P), \ldots, e_d(P))$ arising from simplicial d-polytopes and from general d-polytopes.

The following theorem motivated by commutative-algebraic problems confirmed a conjecture by Kleinschmidt, Kalai and Lee.

THEOREM 19.5.35 *Migliore and Nagel, [81]* (2002)

For all simplicial d-polytopes with prescribed h-vector $h = (h_0, h_1, \ldots, h_d)$, the number of i-dimensional empty simplices is maximized by the Billera-Lee polytopes $P_{BL}(h)$.

$P_{BL}(h)$ is the polytope constructed by Billera and Lee [16] (see Chapter 17) in their proof of the sufficiency part of the g-theorem. Migliore and Nagel proved that for a prescribed f-vector, the Billera-Lee polytopes maximizes even more general parameters that arise in commutative algebra: The sum of the i-th Betti numbers of induced subcomplexes on j vertices for every i and j. (The case $j = i + 2$ reduces to counting missing faces.) It is quite possible that the theorem of Migliore and Nagel extends to general simplicial spheres with prescribed h-vector and to general polytopes with prescribed (toric) h-vector. (However, it is not yet known in these cases that the h-vectors are always those of Billera-Lee polytopes, see Chapter 17.)

19.6 CONCLUDING REMARKS AND EXTENSIONS TO MORE GENERAL OBJECTS

The reader who compares this chapter with other chapters on convex polytopes may notice the sporadic nature of the results and problems described here. Indeed, it seems that our main limits in understanding the combinatorial structure of polytopes still lie in our ability to raise the right questions. Another feature that comes to mind (and is not unique to this area) is the lack of examples, methods of constructing them, and means of classifying them.

We have considered mainly properties of general polytopes and of simple or simplicial polytopes. There are many classes of polytopes that are either of intrinsic interest from the combinatorial theory of polytopes, or that arise in various other fields, for which the problems described in this chapter are interesting.

Most of the results of this chapter extend to much more general objects than convex polytopes. Finding combinatorial settings for which these results hold is an interesting and fruitful area. On the other hand, the results described here are not sufficient to distinguish polytopes from larger classes of polyhedral spheres, and finding delicate combinatorial properties that distinguish polytopes is an important area of research. Few of the results on skeletons of polytopes extend to skeletons of other convex bodies [74, 75, 76], and relating the combinatorial theory of polytopes with other aspects of convexity is a great challenge.

19.7 SOURCES AND RELATED MATERIAL

FURTHER READING

Grünbaum [39] is a survey on polytopal graphs and many results and further references can be found there). More material on the topic of this chapter and further relevant references can also be found in [36], [108], [18], [65], and [14]. Martini's chapter in [18] is on the regularity properties of polytopes (a topic not covered here; cf. Chapter 16), and contains further references on facet-forming polytopes and nonfacets. The original papers on facet-forming polytopes and nonfacets contain many more results, and describe relations to questions on tiling spaces with polyhedra. Other chapters of [18] are also relevant to the topic of this section.

RELATED CHAPTERS

Chapter 15: Basic properties of convex polytopes
Chapter 17: Face numbers of polytopes and complexes
Chapter 44: Linear programming in low dimensions
Chapters 5,16,18,20,45 and 59 are also related to some parts of this chapter.

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