Shapley Value

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The *value* of an uncertain outcome (a 'gamble', 'lottery', etc.) to a participant is an evaluation, in the participant's utility scale, of the prospective outcomes: It is an *a priori* measure of what he expects to obtain (this is the subject of 'utility theory'). In a similar way, one is interested in evaluating a *game*; that is, measuring the *value* of each player in the game.

Such an approach was originally developed by Lloyd S. Shapley (1951, 1953a). The framework was that of *n-person games in coalitional form with side-payments*. Such a game is given by a finite set N together with a function v that associates to every subset S of N a real number $v(S)$. Here, N is the set of 'players', v is the 'characteristic function' and $v(S)$ is the 'worth' of the 'coalition' S: the maximal total payoff the members of S can obtain. This model presumes the following: (i) There are finitely many players. (ii) Agreements between players are possible and enforceable (the game is 'cooperative'). (iii) There exists a medium of exchange ('money') that is freely transferable in unlimited amounts between the players, and moreover every player's utility is additive with respect to it (i.e. a transfer of x units from one player to another decreases the first one's utility by x units and increases the second one's utility by x units; the total payoff of a coalition can thus be meaningfully defined as the sum of the payoffs of its members). This assumption is known as existence of 'side-payments' or 'transferable utility'. (iv) The game is adequately described by its characteristic function (i.e. the worth $v(S)$ of each coalition S is well defined, and the abstraction from the extensive structure of the game to its characteristic function leads to no essential loss; the game is then called a 'c-game'). It should be noted that these underlying assumptions may be interpreted in a broader and more abstract sense. For example, in a voting situation, a 'winning coalition' is assigned worth 1, and a 'losing' coalition – worth 0. The important feature is that, for each coalition, its prospects may well be summarized by one real number.

The *Shapley value* of such a game is a unique payoff vector for the game (i.e. a payoff to each player). It is determined by the following four axioms (this differs from Shapley's original approach only unessentially). (1) *Symmetry* or *equal treatment*: If two players in a game are substitutes (i.e. the worth of no coalition changes when replacing one of the two players by the other one), then their values are equal. (2) *Null* or *dummy player*: If a player in a game is such that the worth of a coalition never changes when he joins it, then his value is zero. (3) *Efficiency* or *Pareto optimality*: The sum of the values of all players equals $v(N)$, the worth of the grand coalition of all players (in a superadditive game $v(N)$ is the maximal amount that the players can jointly get; note that this axiom actually combines feasibility with efficiency). (4) *Additivity*: The value of the sum of two games is the sum of the values of the two games (an equivalent requirement is that the value of a probabilistic combination of two games is the same as the probabilistic combination of the values of the two games; this is analogous to the 'expected utility' postulate). The result of Shapley is that these axioms uniquely determine one payoff vector for each game.

Remarkably, the Shapley value of a player i in a game v turns out to be exactly the *expected marginal contribution of player i to a random coalition S*. For a coalition S not containing i, the marginal contribution to i to S is the change in the worth when i joins S, i.e. $v(S \cup \{i\}) - v(S)$. A random coalition S not containing i is obtained by arranging all n players in line (e.g. $1, 2, \dots, n$), and then putting in S all those that precede i; it is assumed that all $n!$ orders are equally likely. This formula is remarkable, first, since it is a consequence of the very simple and basic axioms above and, second, since the idea of marginal contribution is so fundamental in much of economic analysis.

It should be emphasized that the value of a game is an *a priori* measure – before the game is actually played. Unlike other solution concepts (e.g. the core, von Neumann–Morgenstern solutions, bargaining sets, etc.), it need not yield a 'stable' outcome (the probable final result when the game is played). These final stable outcomes are in general not well determined; the value – which is uniquely specified – may be thought of as their *expectation* or average. Another interpretation of the value axioms regards them as rules for 'fair' division, guiding an impartial 'referee' or 'arbitrator'. Moreover, as suggested above, the Shapley value may be understood as the utility of playing the game (Shapley, 1953a; for a formalization, see Roth, 1977).

In view of both its strong intuitive appeal and its mathematical tractability, the Shapley value has been the focus of much research and applications. Some of these will be briefly mentioned here (an excellent survey is Aumann, 1978).

CHANGING THE DOMAIN. Following Shapley's pioneering approach, the concept of *value* has been extended to additional classes of games, dispensing with (part of) the assumptions (i)–(iv) above.

'Large games' – where the number of players is infinite – have been extensively studied. This includes games with countably many players (Shapley, 1962; Artstein, 1971; Berbee, 1981); non-atomic games (a continuum of small players who are individually insignificant; the monumental book of Aumann and Shapley, 1974; Kannai, 1966; Neyman and Taumann, 1976; Hart, 1977a; Neyman, 1977, 1981; Tauman, 1977; Mertens, 1980); 'oceanic games' (a continuum of small players together with finitely many larger players; Shapiro and Shapley, 1960; Milnor and Shapley, 1961; Hart, 1973; Fogelman and Quinzii, 1980; Neyman, 1986). The study of large games, which involves the solution of deep mathematical problems, has led to very valuable insights. One example is the so-called 'diagonal principle'; the value is determined by those coalitions S which are close in composition to the whole population (i.e. the proportion of each type of player in S is almost the same as in the grand coalition N of all players).

When the game is not necessarily a 'c-game' (see (iv) above; this is the case when, for example, threats by the complement $T = N - S$ of the coalition S are costly to T), Harsanyi (1959) has suggested using a 'modified characteristic function' and applying to it the Shapley value (see also Selten, 1964).

Another class of much interest consists of games 'without side-payments', or 'with non-transferable utility' ('NTU-games', for short); here, assumption (ii) on the existence of a medium of utility exchange is not necessarily satisfied. The simplest such games – two-person pure bargaining – were originally studied by Nash (1950): a unique solution is determined by a set of simple axioms. A value for general NTU-games, which coincides with the Shapley value in the side-payments case, and with the Nash solution in the two-person case, was proposed by Harsanyi (1959, 1963). Another value (with the same properties) was introduced by Shapley (1969). The latter has been widely studied, in particular in large economic models (see below).

Other extensions include games with communication graphs (Myerson, 1977), coalition structures (Aumann and Drèze, 1974; Owen, 1977; Hart and Kurz, 1983), and so on.

CHANGING THE AXIOMS. The four axioms (1) to (4) have been in turn replaced by alternative axioms or even completely dropped. This has led to new foundations for the Shapley value, as well as to the introduction of various generalizations.

If, in addition to the characteristic function, the data of the game include (relative) weights between the players, then a *weighted Shapley value* may be defined (Shapley, 1953b). In the unanimity game, for example, the values of the players are no longer equal but, rather, proportional to the weights; the usual (symmetric) Shapley value results if all the weights are equal. This model is useful when players are of unequal 'size' (e.g. a player may represent a 'group', a 'department', and so on).

Abandoning the efficiency axiom leads to the class of *semi-values* (Dubey, Neyman and Weber, 1981). An interesting semi-value is the *Banzhaf–Coleman index* (Banzhaf, 1965; Dubey and Shapley, 1979); it has been proposed originally as a measure of power in voting games. It can be computed in the same way as the Shapley value: expected marginal contribution, but assuming that all coalitions not containing player i are equally likely.

Another approach to the value uses the following 'consistency' or 'reduced game' property: Given a solution concept (that associates payoff vectors to games), assume that a group of players in a game has already agreed to it, and they are paid off accordingly; consider the reduced game among the remaining players. If the solution of the reduced game is always the same as that of the original game, then the solution is said to be *consistent*. It turns out that consistency, together with some simple requirements for two-player games, characterizes the Shapley value (Hart and Mas-Colell, 1989).

ECONOMIC APPLICATIONS. The model of an *exchange economy* has been the focus of much study in economic theory. The main solution concept there is the *competitive equilibrium*, where prices are determined in such a way that total supply equals total demand. The cooperative game obtained by allowing each coalition to exchange freely the commodities they own among themselves, is called a *market game*. One can then find the value of the corresponding market game. The following result (known as the 'value equivalence principle') has been obtained in various models of this kind (in particular, both when utility is transferable and when it is not): In a *large* exchange economy (where traders are individually insignificant), all value allocations are competitive; moreover, if the utilities are smooth, then all competitive allocations are also value allocations. (It should be noted that in the NTU case there may be more than one value allocation.) This remarkable result joints together two very different approaches; on the one hand, competitive prices which arise from supply and demand; on the other hand, marginal contributions of the economic agents (Shapley, 1964; Shapley and Shubik, 1969; Aumann and Shapley, 1974; Aumann, 1975; Champsaur, 1975; Hart, 1977b; Mas-Colell, 1977; note moreover that for large markets, the set of competitive equilibria coincides with the core).

Other applications of the value to economic theory include models of taxation, where a political power structure is superimposed on the exchange or production economy (Aumann and Kurz, 1977). Further references on economic applications can be found in Aumann (1985).

Next, consider the problem of allocating joint costs in a 'fair' manner. It turns out that the axioms determining the Shapley value are easily translated into postulates suitable for this kind of problem (e.g. the efficiency axiom becomes total cost sharing). The various 'tasks' (or 'projects', 'departments', etc.) become the players, and $v(S)$ is the total cost of the set S of tasks (Shubik, 1962). Two notable applications are airport landing fees (a task here is an aircraft landing; Littlechild and Owen, 1973) and telephone billing (each time unit of a phone call is a player; the resulting cost allocation scheme is in actual use at Cornell University; Billera, Heath and Raanan, 1978). Further research in this direction can be found in Shapley (1981a; the use of weighted Shapley values is proposed there); Billera and Heath (1982); Mirman and Tauman (1982); and the book edited by Young (1985).

POLITICAL APPLICATIONS. The Shapley value has been widely applied to the study of power in voting and other political systems. A trivial observation – although not always remembered in practice – is that the political power need not be proportional to the number of votes (the Board of Supervisors in the Nassau County, N. Y. is a good example; Shapley, 1981b). It is therefore important to find an objective method of measuring power in such situations. The Shapley value (known in this setup as the *Shapley–Shubik index*; Shapley and Shubik, 1954) is, by its definition, a very good candidate. Indeed, consider a simple political game; it is described by specifying for each coalition whether it is 'winning' or 'losing'. The Shapley value of a player i is then the probability that i is a 'pivot'; namely, that in a random order of all players, those preceding i are losing, whereas together with i they are winning. For example, in a simple majority voting situation (half of the votes are needed to win), assume there is one large party having $\frac{1}{3}$ of the votes, and the rest of the votes are divided among many small parties; the value of the large party is then approximately $\frac{1}{2}$ – much higher than its share of the votes. In comparison, when there are two large parties each having $\frac{1}{3}$ of the votes (the rest being again divided among a large number of small parties), the value of each large party is close to $\frac{1}{4}$ – less than its voting weight (this phenomenon may be explained by the competition – or lack of it – for the 'favours' of the small parties).

The Shapley value has also been used in more complex models, where 'ideologies' and 'issues' are taken into account (thus, not all arrangements of the voters are equally likely; an 'extremist' party, for example, is less likely to be the pivot than a 'middle-of-the road' one; Owen, 1971; Shapley, 1977). References on political applications of the Shapley value may be found in Shapley (1981b); these include various parliaments (USA, France, Israel), United Nations Security Council and others.

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